

ALGEBRAIC CYCLES AND CRYSTALLINE COHOMOLOGY

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ABSTRACT. We show that additive higher Chow groups on smooth varieties over a field of characteristic $p \neq 2$ induce a Zariski sheaf of pro-differential graded algebras, whose Milnor range is isomorphic to the Zariski sheaf of the big de Rham-Witt complexes of L. Hesselholt and I. Madsen. When $p > 2$, the Zariski hypercohomology of the p -typical part of the sheaves arising from additive higher Chow groups computes the crystalline cohomology of smooth varieties. This revisits the 1970s results of S. Bloch and L. Illusie on crystalline cohomology.

1. INTRODUCTION

One of the aims of this paper is to build a bridge between two different worlds, where S. Bloch had made fundamental contributions decades ago: at one end is the world of crystalline cohomology of P. Berthelot [3] of smooth varieties over a perfect field k of characteristic $p > 0$. This is computable via parts of Quillen relative higher algebraic K -groups of certain nilpotent schemes as shown in [4]. At the other end is the world of algebraic cycles, in particular higher Chow groups [5] which give the motivic cohomology theory for smooth k -schemes. Sure, the cycle class maps from the motivic cohomology to crystalline cohomology do provide a bridge of certain kind, but what we are after here is not this sort of Hodge-Tate type question. We ask whether there is a way to construct the crystalline cohomology itself from groups arising from algebraic cycles.

This might sound odd to some people, but several works indicate that this is plausible. In the 1970s, L. Illusie [20] proved that the crystalline cohomology is computable as hypercohomology of the p -typical de Rham-Witt complexes that are isomorphic to the p -typical curves of S. Bloch in [4]. The p -typical de Rham-Witt complexes were generalized to the “multi-prime” big de Rham-Witt complexes $\mathbb{W}_m\Omega_X^\bullet$ by L. Hesselholt and I. Madsen in [17] in the 1990s. More recently, in the 2000s, S. Bloch and H. Esnault [6] and K. R ulling [41] proved that for $X = \operatorname{Spec}(k)$, the big de Rham-Witt complex comes from algebraic cycles, via the (cubical) additive higher Chow groups of 0-cycles.

Back then, we did not know much about the additive higher Chow groups, but during the past several years they were developed and studied from various perspectives (see [22], [26], [27], [28], [29], [37], [38]). Based on these recent developments, this paper attempts to go back to the question on describing the big de Rham-Witt complexes from additive higher Chow groups.

2010 *Mathematics Subject Classification.* Primary 14C25; Secondary 13F35, 14F30, 19E15.

Key words and phrases. algebraic cycle, additive higher Chow group, de Rham-Witt complex, generic projection, Gersten conjecture, crystalline cohomology.

Our main result below is a generalization of the theorem of K. R ulling [41] as well as the additive analogue of results of P. Elbaz-Vincent and S. M ller-Stach [10], M. Kerz [24], and M. Kerz and S. M ller-Stach [25]. A good part of it is stated as Theorem 4.2.2:

Theorem 1.0.1. *Let k be any field of characteristic $p \neq 2$. Let R be a regular semi-local k -algebra essentially of finite type. Then for all $n, m \geq 1$, we have isomorphisms between the big de Rham-Witt forms and the additive higher Chow groups*

$$(1.1) \quad \tau_{n,m}^R : \mathbb{W}_m \Omega_R^{n-1} \xrightarrow{\sim} \mathrm{TCH}^n(R, n; m).$$

In particular, the additive higher Chow groups of $\mathrm{Spec}(R)$ in the Milnor range form the universal restricted Witt-complex over R .

Using Theorem 1.0.1, one can identify the Zariski sheaf of big de Rham-Witt complexes with the Zariski sheafification of a presheaf constructed from additive higher Chow groups in the Milnor range. All sections of the paper, other than §13, which discusses some applications, are devoted to proving this theorem. In §2 and §3, we recall the definitions of various objects and some basic results we use. In §4, we recall the construction of the maps $\tau_{n,m}^R$ and show that $\tau_{n,m}^R$ is injective. The remaining surjectivity part of $\tau_{n,m}^R$ is more challenging, and §5 ~ 12 are all about it. We first show in §5 ~ 7 that every class in $\mathrm{TCH}^n(R, n; m)$ can be represented by a cycle whose supports are all finite and surjective over R . If we denote the subgroup of such cycle classes as $\mathrm{TCH}_{\mathrm{fs}}^n(R, n; m)$, then we have $\mathrm{TCH}_{\mathrm{fs}}^n(R, n; m) \xrightarrow{\sim} \mathrm{TCH}^n(R, n; m)$. We go further in §8 ~ 11 to show that the cycle classes in $\mathrm{TCH}_{\mathrm{fs}}^n(R, n; m)$ can be represented by cycles that have additional smoothness properties. The subgroup of such cycle classes is denoted by $\mathrm{TCH}_{\mathrm{sfs}}^n(R, n; m)$. The following, stated as Theorem 5.2.3, summarizes this:

Theorem 1.0.2. *Let k be an arbitrary field and let R be a smooth semi-local k -algebra of geometric type. Then*

$$\mathrm{TCH}_{\mathrm{sfs}}^n(R, n; m) \xrightarrow{\sim} \mathrm{TCH}_{\mathrm{fs}}^n(R, n; m) \xrightarrow{\sim} \mathrm{TCH}^n(R, n; m).$$

Here, “of geometric type” means, as in Definition 2.4.2, it is localized at a finite set of closed points of a finite type k -scheme. Using this and some other results, we can reduce every cycle class of $\mathrm{TCH}^n(R, n; m)$ to a sum of special types of cycles, that we call *Witt-Milnor graph cycles* or *symbolic cycles*, over an extension ring R' over R . Unlike the case of higher Chow groups of regular semi-local k -schemes studied in [10] and [25], where they can use the pre-existing norm maps for Milnor K -groups, this does not yet solve our surjectivity problem, especially due to lack of the construction of the trace maps on the big de Rham-Witt forms for suitable finite extensions of rings. Our approach is to use the map $\tau_{n,m}^R$ and the groups $\mathrm{TCH}^n(R, n; m)$ to define the notion of “traceability” for forms in $\mathbb{W}_m \Omega_{R'}^{n-1}$, where R' is a finite simple ring extension of R . We use the Witt-complex structure of the additive higher Chow groups as proven in [32] to show that all forms in $\mathbb{W}_m \Omega_{R'}^{n-1}$ are traceable. We can then deduce the desired surjectivity of $\tau_{n,m}^R$ using an induction argument. In §13, we discuss applications of Theorem 1.0.1. We list some of them:

Theorem 1.0.3. *Let k be a perfect field of characteristic $\neq 2$. Let R be a regular semi-local k -algebra essentially of finite type. Then the additive higher Chow groups of R in the Milnor range satisfy the Gersten conjecture in the sense that the Cousin complex is a flasque resolution. In particular, when $K = \text{Frac}(R)$, the flat pull-back map $\text{TCH}^n(R, n; m) \rightarrow \text{TCH}^n(K, n; m)$ is injective.*

A combination of Theorem 1.0.1 and [18, Theorem C] yields the following étale descent for the additive higher Chow groups of regular semi-local rings in the Milnor range:

Theorem 1.0.4. *Let k be a field of characteristic $\neq 2$. Let $R \rightarrow R'$ be an étale extension of regular semi-local k -algebras essentially of finite type. Then the map $\mathbb{W}_m(R') \otimes_{\mathbb{W}_m(R)} \text{TCH}^n(R, n; m) \rightarrow \text{TCH}^n(R', n; m)$ is an isomorphism of $\mathbb{W}_m(R')$ -modules for $n, m \geq 1$.*

Using the decomposition of the big de Rham-Witt forms into the direct sum of p -typical de Rham-Witt forms and [20, Proposition I.3.7], we get the following vanishing result for the additive higher Chow groups of high codimension, where the case of $\text{Spec}(k)$ was previously shown in [41]:

Theorem 1.0.5. *Let k be a perfect field of characteristic $p > 2$ and let R be a regular semi-local k -algebra essentially of finite type of dimension d . Then $\text{TCH}^n(R, n; m) = 0$ for $m \geq 1$ and $n > d + 1$. Moreover, the Frobenius map $F : \text{TCH}^{d+1}(R, d+1; \bullet) \rightarrow \text{TCH}^{d+1}(R, d+1; \bullet)$ is an isomorphism of pro-abelian groups. In particular $\text{TCH}^1(k, 1; m) \simeq \mathbb{W}_m(k)$ and $\text{TCH}^n(k, n; m) = 0$ for $n > 1$.*

At the global level, we can use Theorem 1.0.1 to prove the existence of the trace maps on the big de Rham-Witt forms for finite ring extensions of regular k -algebras, by way of push-forwards of additive higher Chow cycles:

Theorem 1.0.6. *Let k be a field of characteristic $\neq 2$. Then for any finite ring extension $R \subset R'$ of regular k -algebras essentially of finite type, there exists a trace map $\text{Tr}_{R'/R} : \mathbb{W}_m \Omega_{R'}^n \rightarrow \mathbb{W}_m \Omega_R^n$, which is transitive, and compatible with the push-forward of additive higher Chow cycles, i.e., the diagram commutes:*

$$(1.2) \quad \begin{array}{ccc} \mathbb{W}_m \Omega_{R'}^{n-1} & \xrightarrow{\tau_{n,m}^{R'}} & \text{TCH}^n(R', n; m) \\ \text{Tr}_{R'/R} \downarrow & & \downarrow f_* \\ \mathbb{W}_m \Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \text{TCH}^n(R, n; m). \end{array}$$

Combining Theorem 1.0.1 with the main theorem of [20], we obtain:

Theorem 1.0.7. *Let X be a smooth scheme of finite type over a perfect field k of characteristic $p > 2$. Then for $n \geq 0$, there is a canonical isomorphism*

$$\text{H}_{\text{crys}}^n(X/W) \simeq \varprojlim_i \mathbb{H}_{\text{Zar}}^{n+1}(X, \mathcal{TCH}_{(p)}^M(-; p^i)_{\text{Zar}}),$$

where $\mathcal{TCH}_{(p)}^M(-; p^i)_{\text{Zar}}$ is the Zariski sheafification of the i -th level of the p -typical part of the additive higher Chow presheaf in the Milnor range as in (3.7).

Since the group on the right hand side originates from objects defined in terms of algebraic cycles, in a sense this isomorphism gives an algebraic-cycle description of the crystalline cohomology groups. In [4], the crystalline cohomology groups were described by the hypercohomology of some complexes of sheaves given in terms of some relative algebraic K -groups, while one of the insights in the development of additive higher Chow groups comes from the desire to describe such relative K -groups in terms of algebraic cycles.

In fact, using Theorem 1.0.2 and some results of [20], it is possible to show that for a smooth projective scheme X over a field of characteristic $p > 2$, there is a canonical map from $\mathcal{TCH}_{(p)}^M(-; p^i)_{\text{Zar}}$ to Bloch's chain complex of Zariski sheaves, which is a quasi-isomorphism if $\dim(X) < p$. This would yield an analogue of Theorem 1.0.7, where $H_{\text{crys}}^*(X/W)$ is replaced by the hypercohomology of Bloch's complex. We wish to come back to it in a different project.

Another consequence of Theorems 1.0.1 and 1.0.7 is that the open Problem 1 of [4, §IV] has a positive solution, if we replace the relative K -theory $C_m^\bullet(R)$ of Bloch by the complex of additive higher Chow groups $\text{TCH}^M(R; m)$.

Conventions. In this paper, a k -scheme is a separated scheme of finite type over k , unless we say otherwise. A k -variety is a reduced k -scheme. The product $X \times Y$ means usually $X \times_k Y$, unless we specify otherwise. We let \mathbf{Sch}_k be the category of k -schemes, \mathbf{Reg}_k of regular k -schemes, \mathbf{Sm}_k of smooth k -schemes, and \mathbf{SmAff}_k of smooth affine k -schemes. A scheme essentially of finite type is a scheme obtained by localizing at a finite subset (including \emptyset) of a finite type k -scheme. For $\mathcal{C} = \mathbf{Sch}_k, \mathbf{Reg}_k, \mathbf{Sm}_k, \mathbf{SmAff}_k$, we let \mathcal{C}^{ess} be the extension of the category \mathcal{C} , whose objects are obtained by localizing an object of \mathcal{C} at a finite subset (including \emptyset). When we say a semi-local k -scheme, we always mean one that is essentially of finite type over k , unless said otherwise.

N.B. In this paper, each section begins with an assumption on the base field k that prevails in the section or in some following sections. However, if a following section declares an exception, then the exception prevails. We tried to minimize such exceptions to avoid potential confusions.

2. BASIC RESULTS ON ADDITIVE HIGHER CHOW GROUPS

Let k be any field. In this section, we recall definitions of higher Chow groups and additive higher Chow groups, and establish some basic results we need.

2.1. Higher Chow groups. We recall (cf. [5], [44]) the definition of higher Chow groups. Let $X \in \mathbf{Sch}_k^{\text{ess}}$ be equidimensional. Let $\mathbb{P}_k^1 = \text{Proj } k[Y_0, Y_1]$, and $\square^n = (\mathbb{P}_k^1 \setminus \{1\})^n$. Let $(y_1, \dots, y_n) \in \square^n$ be the coordinates. A *face* of \square^n is a closed subscheme defined by a set of equations of the form $y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s$, where $\epsilon_j \in \{0, \infty\}$. For $1 \leq i \leq n$ and $\epsilon = 0, \infty$, we let $\iota_i^\epsilon : \square^{n-1} \rightarrow \square^n$ be the closed immersion given by $(y_1, \dots, y_{n-1}) \mapsto (y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-1})$. Its image gives a codimension 1 face.

Let $q, n \geq 0$. Let $\underline{z}^q(X, n)$ be the free abelian group on the set of integral closed subschemes of $X \times \square^n$ of codimension q , that intersect properly with $X \times F$ for each face F of \square^n . We define the boundary map $\partial_i^\epsilon(Z) := [(\text{Id}_X \times \iota_i^\epsilon)^*(Z)]$. This collection of data gives a cubical abelian group $(\underline{n} \mapsto \underline{z}^q(X, n))$ in the sense of

[26, §1.1], and the groups $z^q(X, n) := \underline{z}^q(X, n) / \underline{z}^q(X, n)_{\text{degn}}$ (in the notations of *loc.cit.*) give a complex of abelian groups, whose boundary map at level n is given by $\partial := \sum_{i=1}^n (-1)^i (\partial_i^\infty - \partial_i^0)$. The homology $\text{CH}^q(X, n) := H_n(z^q(X, \bullet), \partial)$ is called the higher Chow group of X .

2.2. Additive higher Chow groups. We recall the definition of additive higher Chow groups from [29, §2]. Let $X \in \mathbf{Sch}_k^{\text{ess}}$ be equidimensional. Let $\mathbb{A}^1 = \text{Spec } k[t]$, $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$, and $\overline{\square} = \mathbb{P}^1$. For $n \geq 1$, let $B_n = \mathbb{A}^1 \times \square^{n-1}$, $\overline{B}_n = \mathbb{A}^1 \times \overline{\square}^{n-1}$ and $\widehat{B}_n = \mathbb{P}^1 \times \overline{\square}^{n-1} \supset \overline{B}_n$. Let $(t, y_1, \dots, y_{n-1}) \in \overline{B}_n$ be the coordinates.

On \overline{B}_n , define the Cartier divisors $F_{n,i}^1 := \{y_i = 1\}$ for $1 \leq i \leq n-1$, $F_{n,0} := \{t = 0\}$, and let $F_n^1 := \sum_{i=1}^{n-1} F_{n,i}^1$. A *face* of B_n is a closed subscheme defined by a set of equations of the form $y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s$ where $\epsilon_j \in \{0, \infty\}$. For $1 \leq i \leq n-1$ and $\epsilon = 0, \infty$, let $\iota_{n,i}^\epsilon: B_{n-1} \rightarrow B_n$ be the inclusion $(t, y_1, \dots, y_{n-2}) \mapsto (t, y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-2})$. Its image is a codimension 1 face.

The additive higher Chow complex is defined similarly using the spaces B_n instead of \square^n , but other than proper intersections with all faces, we impose additional conditions called the *modulus conditions*, that control how the cycles should behave at “infinity”:

Definition 2.2.1 ([29, Definition 2.1]). Let X be a k -scheme, and let V be an integral closed subscheme of $X \times B_n$. Let \overline{V} denote the Zariski closure of V in $X \times \overline{B}_n$ and let $\nu: \overline{V}^N \rightarrow \overline{V} \subset X \times \overline{B}_n$ be the normalization of \overline{V} . Let $m, n \geq 1$ be integers. We say that V satisfies the *modulus m condition* on $X \times B_n$, if as Weil divisors on \overline{V}^N we have $(m+1)[\nu^*(F_{n,0}^1)] \leq [\nu^*(F_n^1)]$. (N.B. When $n = 1$, we have $F_1^1 = \emptyset$, so it means $\nu^*(F_{1,0}^1) = 0$, or $\{t = 0\} \cap \overline{V} = \emptyset$.) If V is a cycle on $X \times B_n$, we say that V satisfies the modulus m condition if each of its irreducible components satisfies the modulus m condition. When m is understood, often we just say that V satisfies the modulus condition. N.B. Since $F_{n,0} = \{t = 0\} \subset \overline{B}_n$, replacing \overline{B}_n by \widehat{B}_n in the definition does not change the nature of the modulus condition on V .

Definition 2.2.2 ([29, Definition 2.5]). Let $X \in \mathbf{Sch}_k^{\text{ess}}$, and let $r, m, n \in \mathbb{Z}$ with $m, n \geq 1$.

- (0) $\underline{\text{Tz}}_r(X, 1; m)$ is the free abelian group on integral closed subschemes Z of $X \times \mathbb{A}^1$ of dimension r , satisfying the modulus condition.

For $n > 1$, $\underline{\text{Tz}}_r(X, n; m)$ is the free abelian group on integral closed subschemes Z of $X \times B_n$ of dimension $r + n - 1$ such that:

- (1) For each face F of B_n , Z intersects $X \times F$ properly on $X \times B_n$.
- (2) Z satisfies the modulus m condition on $X \times B_n$.

For each $1 \leq i \leq n-1$ and $\epsilon = 0, \infty$, let $\partial_i^\epsilon(Z) := [(\text{Id}_X \times \iota_{n,i}^\epsilon)^*(Z)]$. The proper intersection with faces ensures that $\partial_i^\epsilon(Z)$ are well-defined.

The cycles in $\underline{\text{Tz}}_r(X, n; m)$ are called the *admissible cycles* (or, often as *additive higher Chow cycles*, or *additive cycles*). When the scheme X is equidimensional of dimension d over k , we write for $q \geq 0$, $\underline{\text{Tz}}^q(X, n; m) := \underline{\text{Tz}}_{d+1-q}(X, n; m)$.

This gives the cubical abelian group $(\underline{n} \mapsto \underline{\mathrm{Tz}}^q(X, n+1; m))$ in the sense of [26, §1.1]. Using the containment lemma [27, Proposition 2.4], that each face $\partial_i^e(Z)$ lies in $\underline{\mathrm{Tz}}_r(X, n-1; m)$ is implied from (1) and (2).

Definition 2.2.3 ([29, Definition 2.6]). Let $X \in \mathbf{Sch}_k^{\mathrm{ess}}$ be equidimensional. The *additive higher Chow complex*, or just the *additive cycle complex*, $\mathrm{Tz}^q(X, \bullet; m)$ of X in codimension q with modulus m is the nondegenerate complex associated to the cubical abelian group $(\underline{n} \mapsto \underline{\mathrm{Tz}}^q(X, n+1; m))$, i.e., $\mathrm{Tz}^q(X, n; m)$ is the quotient $\underline{\mathrm{Tz}}^q(X, n; m) / \underline{\mathrm{Tz}}^q(X, n; m)_{\mathrm{degn}}$.

The boundary map of this complex at level n is given by $\partial := \sum_{i=1}^{n-1} (-1)^i (\partial_i^\infty - \partial_i^0)$, and it satisfies $\partial^2 = 0$. The homology $\mathrm{TCH}^q(X, n; m) := H_n(\mathrm{Tz}^q(X, \bullet; m))$ for $n \geq 1$ is the *additive higher Chow group* of X with modulus m .

2.3. Subcomplexes associated to some algebraic subsets. Let $X \in \mathbf{Sch}_k^{\mathrm{ess}}$ be equidimensional. Here are some subgroups of $\mathrm{Tz}^q(X, n)$ with a finer intersection property with a given finite set \mathcal{W} of locally closed algebraic subsets of X :

Definition 2.3.1 (cf. [27, Definition 4.2]). Define $\underline{\mathrm{Tz}}_{\mathcal{W}}^q(X, n; m)$ to be the subgroup of $\underline{\mathrm{Tz}}^q(X, n; m)$ generated by integral closed subschemes $Z \subset X \times B_n$ that additionally satisfy

$$(2.1) \quad \mathrm{codim}_{W \times F}(Z \cap (W \times F)) \geq q \text{ for all } W \in \mathcal{W} \text{ and all faces } F \subset B_n.$$

The groups $\underline{\mathrm{Tz}}_{\mathcal{W}}^q(X, n+1; m)$ for $n \geq 0$ form a cubical subgroup of $(n \mapsto \underline{\mathrm{Tz}}^q(X, n+1; m))$ and they give the subcomplex $\mathrm{Tz}_{\mathcal{W}}^q(X, \bullet; m) \subset \mathrm{Tz}^q(X, \bullet; m)$ by modding out by the degenerate cycles. The homology groups are denoted by $\mathrm{TCH}_{\mathcal{W}}^q(X, n; m)$.

2.4. Some properties. Recall that (cf. [12, §2.2]) we say a scheme X is an FA-scheme if given any finite subset $\Sigma \subset X$, there exists an affine open subset $U \subset X$ such that $\Sigma \subset U$. We have the following (*loc.cit.*):

Lemma 2.4.1. *Any quasi-projective k -scheme is FA. Any open subset of an FA-scheme is FA. Given any finite subset Σ of a quasi-projective k -scheme, and an open subset $U \subset X$ containing Σ , there exists an affine open subset $W \subset U$ containing Σ .*

Definition 2.4.2. Recall a semi-local k -algebra R is *essentially of finite type* if there is a connected quasi-projective k -scheme $X = \mathrm{Spec}(A)$ of finite type over k and a finite set of (not necessarily closed) points Σ of X such that $R = \mathcal{O}_{X, \Sigma}$. We say that it is *of geometric type* if Σ consists of only closed points. The pair (X, Σ) will be called an *atlas for $V = \mathrm{Spec}(R)$* . An *affine open subatlas* (Y, Σ) of (X, Σ) for V is an atlas for V such that $Y \subset X$ is an affine open subset.

By Lemma 2.4.1, for any semi-local k -algebra R essentially of finite type, we always have an atlas (U, Σ) , where U is affine. We recall the following result from [32, Lemmas 4.13, 4.14] used in this paper:

Lemma 2.4.3. *Let $V = \mathrm{Spec}(R)$ be a semi-local k -scheme essentially of finite type with a finite set of points Σ . Let $m, n, q \geq 1$ and let $\alpha \in \mathrm{Tz}^q(V, n; m)$ be a cycle. Then there exists an atlas (X, Σ) for V and a cycle $\bar{\alpha} \in \mathrm{Tz}^q(X, n; m)$ such that $\bar{\alpha}_V = \alpha$. If $\partial(\alpha) = 0$, we can assume that $\partial(\bar{\alpha}) = 0$. If $\alpha \in \mathrm{Tz}_{\Sigma}^q(V, n; m)$, then $\bar{\alpha}$ is necessarily in $\mathrm{Tz}_{\Sigma}^q(X, n; m)$.*

The following results will be used later to deal with the case when k is an imperfect field. Recall from [27, Lemma 2.2]:

Lemma 2.4.4. *Let $f : Y \rightarrow X$ be a surjective map of normal integral k -schemes. Let D be a Cartier divisor on X such that $f^*(D) \geq 0$ on Y . Then $D \geq 0$ on X .*

Lemma 2.4.5. *Let $k \hookrightarrow K$ be an extension of fields. Let $\lambda : R \rightarrow R'$ be a faithfully flat morphism of noetherian k -algebras such that $K \hookrightarrow R'$. Let $X = \operatorname{Spec}(R)$ and $X' = \operatorname{Spec}(R')$. Then the induced map of schemes $X' \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1} \rightarrow X \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}$ is also faithfully flat.*

Proof. Since the statement of the lemma is local on $X \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}$, we can replace $\overline{\square}_k^{n-1} \simeq \mathbb{P}_k^{n-1}$ by \mathbb{A}_k^{n-1} . The problem then reduces to showing that the map $R[t, y_1, \dots, y_{n-1}] \rightarrow R'[t, y_1, \dots, y_{n-1}]$ is faithfully flat. This is obvious because $R \rightarrow R'$ is faithfully flat. \square

Lemma 2.4.6. *Let $k \hookrightarrow K$ be a field extension, not necessarily finitely generated, and let A be a K -algebra essentially of finite type. Let $\{A_i\}_{i \geq 1}$ be a direct system of k -algebras essentially of finite type such that each map $\lambda_i : A_i \rightarrow A_{i+1}$ is faithfully flat, and we have $\varinjlim_i A_i = A$. Then the flat pull-back of additive higher Chow groups induces an isomorphism $\varinjlim_i \operatorname{TCH}^q(A_i, n; m) \xrightarrow{\sim} \operatorname{TCH}^q(A, n; m)$.*

Proof. Since the homology functor commutes with the direct limit, it suffices to show that the lemma holds at the level of cycles complexes. We set $V_i = \operatorname{Spec}(A_i)$ and $V = \operatorname{Spec}(A)$. Let $\lambda'_i : A_i \rightarrow A$ be the natural map. Notice that λ'_i is also faithfully flat. This can be easily checked using the fact that a direct limit of flat modules is flat, and an A_i -module M is faithfully flat if and only if it is flat and $\mathfrak{m}M \neq 0$ for every nonzero maximal ideal $\mathfrak{m} \subset A_i$ (see [36, Theorem 7.2]).

The injectivity of the map $\varinjlim_i \operatorname{Tz}^q(V_i, \bullet; m) \rightarrow \operatorname{Tz}^q(V, \bullet; m)$ is obvious. To show its surjectivity, let $Z \in \operatorname{Tz}^q(V, n; m)$ be an irreducible admissible cycle and let $\overline{Z} \subset V \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1}$ be its Zariski closure and $\nu_Z : \overline{Z}^N \rightarrow V \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1}$ the normalization map. Since each of $f_i := \operatorname{Spec}(\lambda_i) : V_{i+1} \rightarrow V_i$ and $f'_i := \operatorname{Spec}(\lambda'_i) : V \rightarrow V_i$ is faithfully flat, by Lemma 2.4.5, the product maps $\tilde{f}_i : V_{i+1} \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1} \rightarrow V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}$ and $\tilde{f}'_i : V \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1} \rightarrow V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}$ are faithfully flat maps of noetherian k -schemes.

Since $V = \varprojlim_i V_i$, it follows from the above faithfully flatness that there exists $i \gg 0$ and an irreducible cycle $\overline{Z}_i \hookrightarrow V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}$ such that $(\tilde{f}'_i)^*(\overline{Z}_i) = \overline{Z}$ and $Z_i := \overline{Z}_i \cap (V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1}) \in z^q(V_i \times_k \mathbb{A}_k^1, n-1)$.

Since the right square in the commutative diagram below

$$(2.2) \quad \begin{array}{ccccccc} \overline{Z}^N & \xrightarrow{\quad \nu_Z \quad} & \overline{Z} & \longrightarrow & V \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1} & \longleftarrow & V \times_K \mathbb{A}_K^1 \times_K \overline{\square}_K^{n-1} \\ \downarrow & & \downarrow & & \downarrow \tilde{f}'_i & & \downarrow \\ \overline{Z}_i^N & \xrightarrow{\quad \nu_{Z_i} \quad} & \overline{Z}_i & \longrightarrow & V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1} & \longleftarrow & V_i \times_k \mathbb{A}_k^1 \times_k \overline{\square}_k^{n-1} \end{array}$$

is Cartesian, we must have $Z = (\tilde{f}'_i)^*(Z_i)$.

Since \widetilde{f}'_i is faithfully flat, it follows that $\overline{Z} = (\widetilde{f}'_i)^*(\overline{Z}_i) \rightarrow \overline{Z}_i$ is also faithfully flat. In particular, it is surjective. We conclude that the induced map $\overline{Z}^N \rightarrow \overline{Z}_i^N$ on the normalizations is also surjective. Since \overline{Z} satisfies the modulus condition, we can now apply Lemma 2.4.4 to conclude that Z_i too satisfies the modulus condition and hence it is in $\mathrm{Tz}^q(V_i, n; m)$. This finishes the proof. \square

We use the following later:

Proposition 2.4.7. *Let R be a smooth k -algebra essentially of finite type. Then there is a natural cap product map*

$$\cap_R : \mathrm{TCH}^q(R, n; m) \otimes_{\mathbb{Z}} \mathrm{CH}^p(R, n') \rightarrow \mathrm{TCH}^{p+q}(R, n + n'; m),$$

that commutes with the pull-back and push-forward whenever they exist. This is given by $a \cap_R b = \Delta_R^(a \times \zeta(b))$, where Δ_R^* is the pull-back via the diagonal map $\Delta_R : \mathrm{Spec}(R) \rightarrow \mathrm{Spec}(R) \times \mathrm{Spec}(R)$.*

Proof. This is an immediate consequence of [31, Theorem 3.12]. \square

3. MILNOR K -GROUPS AND DE RHAM-WITT COMPLEXES

Let k be any field. In this section, we recall the definitions of Milnor K -groups, rings of big Witt vectors, and big de Rham-Witt complexes, and discuss some properties we need.

3.1. Milnor K -groups. Let R be a commutative ring with unity. Recall that the Milnor K -ring $K_*^M(R)$ of R is the quotient $T_{\mathbb{Z}}^*(R^\times)/I$ of the tensor algebra by the two-sided ideal I generated by $\{a \otimes (1 - a) \mid a \in R^\times, 1 - a \in R^\times\}$. Its degree n -part is $K_n^M(R)$ and $\{a_1, \dots, a_n\}$ is the image of $a_1 \otimes \dots \otimes a_n$, where $a_i \in R^\times$.

3.2. Rings of Witt-vectors. Let R be a commutative ring with unity. Recall the definition of the rings of big Witt-vectors of R from [41, Appendix A]. A *truncation set* $S \subset \mathbb{N}$ is a nonempty subset such that if $s \in S$ and $t|s$, then $t \in S$. As a set, let $\mathbb{W}_S(R) := R^S$ and define the map $w : \mathbb{W}_S(R) \rightarrow R^S$, $a = (a_s)_{s \in S} \mapsto w(a) = (w(a)_s)_{s \in S}$, where $w(a)_s := \sum_{t|s} ta_t^{\frac{s}{t}}$. When R^S on the target of w is given the component-wise ring structure, it is known that there is a unique functorial ring structure on $\mathbb{W}_S(R)$ such that w is a ring homomorphism. See [18, Proposition 1.2]. For two truncation sets $S \subset S'$, there is a restriction $\mathfrak{R} : \mathbb{W}_{S'}(R) \rightarrow \mathbb{W}_S(R)$. When $S = \{1, \dots, m\}$, we write $\mathbb{W}_m(R) = \mathbb{W}_S(R)$. For a fixed prime number p , when $S = \{1, p, p^2, \dots\}$ and $S_i = \{1, p, \dots, p^{i-1}\}$, we write $W(R) = \mathbb{W}_S(R)$ and $W_i(R) = \mathbb{W}_{S_i}(R)$. They are the p -typical rings of Witt vectors.

There is another description of the ring $\mathbb{W}_S(R)$. Let $\mathbb{W}(R) := \mathbb{W}_{\mathbb{N}}(R)$. There is a natural bijection $\mathbb{W}(R) \simeq (1 + TR[[T]])^\times$, where T is an indeterminate and the addition of the ring $\mathbb{W}(R)$ corresponds to the multiplication of the formal power series. For a truncation set S , we have $\mathbb{W}_S(R) = (1 + TR[[T]])^\times / I_S$ for a suitable subgroup I_S . See [41, A.7] for details. In case $S = \{1, \dots, m\}$, we have an isomorphism

$$(3.1) \quad \gamma : \mathbb{W}_m(R) \simeq (1 + TR[[T]])^\times / (1 + T^{m+1}R[[T]])^\times; \quad (a_i)_{1 \leq i \leq m} \mapsto \prod_{i=1}^m (1 - a_i T^i)$$

There is the Teichmüller lift map $[-]_S : R \rightarrow \mathbb{W}_S(R)$ given by $a \mapsto 1 - aT$, which is a multiplicative map. If $S = \{1, \dots, m\}$, we write $[-]_m$ for $[-]_S$. For each $i \geq 1$, we have the i -th Verschiebung V_i given by $V_i([a]_{[m/i]}) = (1 - aT^i)$, where for a non-negative real number x , one denotes by $[x]$ the greatest integer not bigger than x . By [41, Properties A.4(i)], for $x = (x_i) \in \mathbb{W}_m(R)$, we have

$$(3.2) \quad x = \sum_{i=1}^m V_i([x_i]_{[m/i]}).$$

3.3. De Rham-Witt complexes. Let R be a $\mathbb{Z}_{(p)}$ -algebra for a prime p . (N.B. If R is a K -algebra for a field K , whether its characteristic is 0 or positive, this is always a $\mathbb{Z}_{(p)}$ -algebra for some p .) For each truncation set S , there is a differential graded algebra $\mathbb{W}_S \Omega_R^\bullet$, called the big de Rham-Witt complex over R , which defines a functor on the category of truncation sets. This is an initial object in the category of V -complex and in the category of Witt-complexes over R . For its rigorous definition, see [17], [18], or [41]. In case S is a finite truncation set, we have $\mathbb{W}_S \Omega_R^\bullet = \Omega_{\mathbb{W}_S(R)/\mathbb{Z}}^\bullet / N_S^\bullet$, where N_S^\bullet is a differential graded ideal given by some generators. See [41, Proposition 1.2] for details. In case $S = \{1, 2, \dots, m\}$, we write $\mathbb{W}_m \Omega_R^\bullet$ for this object. For a prime number p , when $S = \{1, p, p^2, \dots\}$ and $S_i = \{1, p, \dots, p^{i-1}\}$, we write $W \Omega_R^\bullet = \mathbb{W}_S \Omega_R^\bullet$ and $W_i \Omega_R^\bullet = \mathbb{W}_{S_i} \Omega_R^\bullet$. They are the p -typical de Rham-Witt complexes.

We recall from [17, Definition 1.1.1] that a *restricted Witt-complex over R* is a pro-system of differential graded \mathbb{Z} -algebras $((E_m)_{m \in \mathbb{N}}, \mathfrak{R} : E_{m+1} \rightarrow E_m)$, together with families of homomorphisms of graded rings $(F_r : E_{rm+r-1} \rightarrow E_m)_{m,r \in \mathbb{N}}$ called Frobenius maps, and homomorphisms of graded groups $(V_r : E_m \rightarrow E_{rm+r-1})_{m,r \in \mathbb{N}}$ called Verschiebung maps, satisfying the following relations for all $n, r, s \in \mathbb{N}$:

- (i) $\mathfrak{R}F_r = F_r \mathfrak{R}$, $\mathfrak{R}V_r = V_r \mathfrak{R}$, $F_1 = V_1 = \text{Id}$, $F_r F_s = F_{rs}$, $V_r V_s = V_{rs}$;
- (ii) $F_r V_r = r$. When $(r, s) = 1$, then $F_r V_s = V_s F_r$ on E_{rm+r-1} ;
- (iii) $V_r(F_r(x)y) = xV_r(y)$ for all $x \in E_{rm+r-1}$ and $y \in E_m$; (projection formula)
- (iv) $F_r dV_r = d$ (where d is the differential of the DGAs).

Furthermore, there is a homomorphism of pro-rings $(\lambda : \mathbb{W}_m(R) \rightarrow E_m^0)_{m \in \mathbb{N}}$ that commutes with F_r and V_r , and we have

- (v) $F_r d\lambda([a]) = \lambda([a]^{r-1})d\lambda([a])$ for all $a \in R$ and $r \in \mathbb{N}$,

where $[a]$ is the Teichmüller lift in $\mathbb{W}_m(R)$ of $a \in R$.

The system $\{\mathbb{W}_m \Omega_R^\bullet\}_{m \geq 1}$ is the initial object in the category of restricted Witt-complexes over R . See [41, Proposition 1.15].

3.4. The presheaf $\mathcal{TC}\mathcal{H}$. Recall from [32] that the additive higher Chow groups $T_{n,m}^q(X) = \text{TCH}^q(X, n; m)$ give a presheaf on \mathbf{SmAff}_k , by using [22]. We have further extensions of these presheaves on \mathbf{Sch}_k defined as follows, motivated while working on [30]:

Definition 3.4.1 ([32, §4.4]). Let $X \in \mathbf{Sch}_k$. The functor $T_{n,m}^q : \mathbf{SmAff}_k^{\text{op}} \rightarrow (\mathbf{Ab})$ induces the functor $T_{n,m}^q : (X \downarrow \mathbf{SmAff}_k)^{\text{op}} \rightarrow (\mathbf{Ab})$. Here, $(X \downarrow \mathbf{SmAff}_k)$ is the category whose objects are the k -morphisms $X \rightarrow A$, with $A \in \mathbf{SmAff}_k$, and a morphism from $h_1 : X \rightarrow A$ to $h_2 : X \rightarrow B$, with $A, B \in \mathbf{SmAff}_k$ is given

by a k -morphism $g : A \rightarrow B$ such that $g \circ h_1 = h_2$. It is cofiltered. Define

$$(3.3) \quad \mathcal{TCH}^q(X, n; m) := \operatorname{colim}_{A \in (X \downarrow \mathbf{SmAff}_k)^{\text{op}}} T_{n,m}^q(A).$$

By [32, Proposition 4.8], we know $\mathcal{TCH}^q(-, n; m)$ is a presheaf on \mathbf{Sm}_k and \mathbf{Sch}_k . In particular, the same holds for $\mathcal{TCH}_{(p)}^n(-; p^i)$. There is a natural homomorphism $\alpha_X : \mathcal{TCH}^q(X, n; m) \rightarrow \text{TCH}^q(X, n; m)$, that becomes an isomorphism for $X \in \mathbf{SmAff}_k^{\text{ess}}$.

For $X \in \mathbf{Sch}_k$, one defines $\mathbb{W}_m \Omega_X^n$ exactly as we defined $\mathcal{TCH}^q(X, n; m)$ in (3.3). For each $\text{Spec}(R) \in \mathbf{SmAff}_k^{\text{ess}}$, we have a natural homomorphism $\tau_{n,m} : \mathbb{W}_m \Omega_R^{n-1} \rightarrow \text{TCH}^n(R, n; m)$ of restricted Witt-complexes (see [32, Theorem 7.11]). So, by taking the colimits over $\operatorname{colim}_{(X \downarrow \mathbf{SmAff}_k)^{\text{op}}}$ and the Zariski sheafifications on \mathbf{Sch}_k , we obtain a morphism of Zariski sheaves $\tau_{n,m} : \mathbb{W}_m \Omega_{(-)}^{n-1} \rightarrow \mathcal{TCH}^n(-, n; m)_{\text{Zar}}$ on \mathbf{Sch}_k , in particular on its subcategory \mathbf{Reg}_k . The main theorem of the paper proven later implies that this map induces a stalk-wise isomorphism, so that the map $\tau_{n,m}$ is an isomorphism of Zariski sheaves on \mathbf{Reg}_k .

3.5. p -typical additive higher Chow groups. Let k be a perfect field of characteristic $p > 2$. Let R be a regular k -algebra essentially of finite type. We know from [32, Theorem 1.2] that $\{\text{TCH}^n(R, n; m)\}_{n,m \geq 1}$ has the structure of a restricted Witt-complex over R . We set $\text{TCH}^n(R; \infty) = \varprojlim_{m \geq 1} \text{TCH}^n(R, n; m)$ and $\text{TCH}^M(R; \infty) = \bigoplus_{n \geq 1} \text{TCH}^n(R; \infty)$. Then $\text{TCH}^M(R; \infty)$ is a differential graded $\mathbb{W}(R)$ -algebra with the compatible Frobenius and Verschiebung operators induced from those on $\{\text{TCH}^n(R, n; m)\}_{n,m \geq 1}$.

Recall from [4, §I.3] that there is a projection $\mathbb{W}(R) \rightarrow W(R)$, where $W(R)$ is the p -typical ring of Witt vectors, given by

$$(3.4) \quad \pi := \sum_{n \in I(p)} \frac{\mu(n)}{n} V_n F_n,$$

where $I(p)$ is the set of integers ≥ 1 not divisible by p , and $\mu(n)$ is the Möbius function. The projection π gives an idempotent element $e = \sum_{n \in I(p)} \frac{\mu(n)}{n} V_n F_n(1) \in \mathbb{W}(R)$. In particular, every $\mathbb{W}(R)$ -module N has a canonical decomposition $N = eN \oplus (1-e)N$. Applying this to $\text{TCH}^n(R; \infty)$, we obtain the *p -typical additive higher Chow groups*

$$(3.5) \quad \text{TCH}_{(p)}^n(R; p^\infty) := e(\text{TCH}^n(R; \infty)), \quad \text{TCH}_{(p)}^M(R; p^\infty) = \bigoplus_{n \geq 1} \text{TCH}_{(p)}^n(R; p^\infty),$$

$$(3.6) \quad \text{TCH}_{(p)}^n(R; p^i) := \text{TCH}_{(p)}^n(R; p^\infty)/p^i, \quad \text{TCH}_{(p)}^M(R; p^i) = \text{TCH}_{(p)}^M(R; p^\infty)/p^i$$

where $i \geq 0$. It follows that each $\text{TCH}_{(p)}^n(R; p^\infty)$ is an $W(R)$ -module. Since $\text{TCH}^q(X, n; m)$ is a presheaf on \mathbf{SmAff}_k , in particular $T_{(p)}^n(X) = \text{TCH}_{(p)}^n(X; p^i)$ gives presheaves on \mathbf{SmAff}_k for $0 \leq i \leq \infty$. So, as in (3.3), we define for $X \in \mathbf{Sch}_k$,

$$(3.7) \quad \mathcal{TCH}_{(p)}^n(X; p^i) := \operatorname{colim}_{A \in (X \downarrow \mathbf{SmAff}_k)^{\text{op}}} \text{TCH}_{(p)}^n(A; p^i).$$

We may also define $W_i\Omega_X^n$ exactly as in (3.3) using the category $(X \downarrow \mathbf{SmAff}_k)^{\text{op}}$. Here, $W_i\Omega_R^n$ and $W\Omega_R^n = \varprojlim_i W_i\Omega_R^n$ are the p -typical de Rham-Witt forms of Deligne and Illusie [20].

We use the p -typical additive higher Chow groups in §13.4 to present crystalline cohomology in terms of additive higher Chow groups.

4. INJECTIVITY OF THE DE RHAM-WITT-CHOW HOMOMORPHISM

In this section, we let k be a perfect field of characteristic $\neq 2$, unless we say otherwise.

4.1. Functoriality. Recall the following functoriality result:

Proposition 4.1.1 ([32, Theorem 7.1, Proposition 7.3]). *Let $f^\sharp : R \rightarrow R'$ be a finite map of regular k -algebras essentially of finite type and let $f : \text{Spec}(R') \rightarrow \text{Spec}(R)$ denote the induced map. Let $n, m, r \geq 1$ be integers. Then, the pull-back $f^* : \text{TCH}^n(R, n; m) \rightarrow \text{TCH}^n(R', n; m)$ and the push-forward $f_* : \text{TCH}^n(R', n; m) \rightarrow \text{TCH}^n(R, n; m)$ satisfy*

$$\begin{aligned} \mathfrak{R}f^* &= f^*\mathfrak{R}; \quad \delta f^* = f^*\delta; \quad F_r f^* = f^*F_r; \quad V_r f^* = f^*V_r; \\ \mathfrak{R}f_* &= f_*\mathfrak{R}; \quad \delta f_* = f_*\delta; \quad F_r f_* = f_*F_r; \quad V_r f_* = f_*V_r. \end{aligned}$$

We discuss the following property of $\tau_{n,m}$.

Lemma 4.1.2. *Let $f^\sharp : R \rightarrow R'$ be a morphism of regular k -algebras essentially of finite type and let $f : \text{Spec}(R') \rightarrow \text{Spec}(R)$ be the induced map. Then $f^* \circ \tau_{1,m}^R([a]) = \tau_{1,m}^{R'} \circ f^*([a])$ for all $a \in R$.*

Proof. Let $\Gamma_{(1-at)}$ denote the cycle in $\text{TCH}^1(R, 1; m)$ corresponding to the ideal $(1-at) \subset R[t]$. By the definition of $\tau_{1,m}^R = \tau_R$, we have $f^* \circ \tau_{1,m}^R([a]) = f^*([\Gamma_{(1-at)}])$. Since $f^*([\Gamma_{(1-at)}]) = [\Gamma_{(1-f^\sharp(a)t)}]$ is an admissible cycle on $\text{Spec}(R) \times \square$, we conclude that $f^*([\Gamma_{(1-at)}]) = \tau_{1,m}^{R'}([f^\sharp(a)])$, which proves the lemma. \square

Set $\text{TCH}(R; m) = \bigoplus_{p,q \geq 0} \text{TCH}^q(R, p; m)$. Recall from [32, Theorem 1.2] that when k is a perfect field of characteristic $\neq 2$, $\text{TCH}(R; m)$ forms a restricted Witt-complex over R . We furthermore have:

Theorem 4.1.3. *Let k be a perfect field of characteristic $\neq 2$. Let $f^\sharp : R \rightarrow R'$ be a morphism of regular k -algebras essentially of finite type and let $f : \text{Spec}(R') \rightarrow \text{Spec}(R)$ be the induced map. Then $f^* : \text{TCH}(R; m) \rightarrow \text{TCH}(R'; m)$ is a morphism of restricted Witt-complexes over R .*

Proof. The theorem is equivalent to prove the following.

- (1) f^* commutes with the products. (2) f^* commutes with the differentials.
- (3) f^* commutes with \mathfrak{R} , F_r and V_r . (4) f^* commutes with $\lambda_R = \tau_R$.

The item (3) is shown in [32, Theorem 7.1] and (4) is shown in Lemma 4.1.2. We need to show (1) and (2).

Set $X = \text{Spec}(R)$ and $X' = \text{Spec}(R')$. To prove (1), it suffices to prove it for irreducible cycles. By [27, Proof of Theorem 7.1], there exists a finite set \mathcal{W} of locally closed subsets of X such that the map $f^* : \text{Tz}_{\mathcal{W}}^q(X, \bullet; m) \rightarrow \text{Tz}^q(X', \bullet; m)$, given by $f^*([Z]) = [f^{-1}(Z)]$, is well-defined. Here, the group on the left of f^* is

as in Definition 2.3.1. Choose an irreducible cycle $[Z] \in \text{Tz}_{\mathcal{W}}^q(X, \bullet; m)$. We claim that there is a finite set \mathcal{C} of locally closed subsets of X such that the following hold.

- (i) $f^* : \text{Tz}_{\mathcal{C}}^q(X, \bullet; m) \rightarrow \text{Tz}^q(X', \bullet; m)$ is defined.
- (ii) $[Z] \boxtimes [Z'] \in \text{Tz}_{\{\Delta_X\}}^q(X \times X, \bullet; m)$ for all $[Z'] \in \text{Tz}_{\mathcal{C}}^q(X, \bullet; m)$.
- (iii) $f^*([Z]) \boxtimes f^*([Z']) \in \text{Tz}_{\{\Delta_{X'}\}}^q(X' \times X', \bullet; m)$ for all $[Z'] \in \text{Tz}_{\mathcal{C}}^q(X, \bullet; m)$.

Let Z_f be the (finite) collection $\{Z_i \cap (X' \times \mathbb{A}^1 \times F)\}$, where Z_i is an irreducible component of $f^*([Z])$ and $F \subset \square^{n-1}$ is a face. It follows from [31, Lemmas 3.5, 3.10] that there exists a finite collection \mathcal{W}' of locally closed subsets of $\{X \times \mathbb{A}^1 \times \square^{n_i}\}$ such that for $\text{Tz}_{\mathcal{W}'}^q(X, \bullet; m)$ in the sense of [28, Definition 5.3], we have

- (i) $f^* : \text{Tz}_{\mathcal{W}'}^q(X, \bullet; m) \rightarrow \text{Tz}_{Z_f}^q(X', \bullet; m)$ is well-defined.
- (ii) $[Z] \boxtimes [Z'] \in \text{Tz}_{\{\Delta_X\}}^q(X \times X, \bullet; m)$ for all $[Z'] \in \text{Tz}_{\mathcal{W}'}^q(X, \bullet; m)$.
- (iii) $Z_i \boxtimes f^*([Z']) \in \text{Tz}_{\{\Delta_{X'}\}}^q(X' \times X', \bullet; m)$ for all $[Z'] \in \text{Tz}_{\mathcal{W}'}^q(X, \bullet; m)$ and all irreducible components Z_i of $f^*([Z])$.

Furthermore, it follows from [31, Lemma 3.4] that there exists a finite collection \mathcal{C}' of locally closed subsets of X such that $\text{Tz}_{\mathcal{W}'}^q(X, \bullet; m) = \text{Tz}_{\mathcal{C}'}^q(X, \bullet; m)$. Setting $\mathcal{C} = \mathcal{W} \cup \mathcal{C}'$, we get the proof of the claim.

We now consider the commutative diagram

$$(4.1) \quad \begin{array}{ccccc} X' \times \mathbb{A}^1 & \xleftarrow{\mu} & X' \times \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{\Delta_{X'}} & X' \times X' \times \mathbb{A}^1 \times \mathbb{A}^1 \\ f \downarrow & & f \downarrow & & \downarrow f \times f \\ X \times \mathbb{A}^1 & \xleftarrow{\mu} & X \times \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{\Delta_X} & X \times X \times \mathbb{A}^1 \times \mathbb{A}^1. \end{array}$$

If we choose any irreducible cycle $Z' \in \text{Tz}_{\mathcal{C}}^q(X, \bullet; m)$, then by the claim $\Delta_X^*([Z] \boxtimes [Z'])$ and $\Delta_X^* \circ (f \times f)^*([Z] \boxtimes [Z'])$ are admissible cycles, and the commutativity of the right square in (4.1) shows that $\Delta_X^* \circ (f \times f)^*([Z] \boxtimes [Z']) = f^*(\Delta_X^*([Z] \boxtimes [Z']))$. In particular, $f^* \circ \Delta_X^*([Z] \boxtimes [Z'])$ is an admissible cycle.

It follows now from [32, Corollary 5.10] that $\mu_* \circ \Delta_X^* \circ (f \times f)^*([Z] \boxtimes [Z'])$ and $\mu_* \circ \Delta_X^* \circ (f \times f)^*([Z] \boxtimes [Z'])$ are admissible cycles. Since the left square in (4.1) is transverse, we get

$$\begin{aligned} f^*([Z] \cdot [Z']) &= f^* \circ \mu_* \circ \Delta_X^*([Z] \boxtimes [Z']) \\ &= \mu_* \circ \Delta_X^* \circ (f \times f)^*([Z] \boxtimes [Z']) = f^*([Z]) \cdot f^*([Z']). \end{aligned}$$

Finally, by [32, Theorem 4.10], the inclusions $\text{Tz}_{\mathcal{C}}^q(X, \bullet; m) \hookrightarrow \text{Tz}^q(X, \bullet; m)$ and $\text{Tz}_{\mathcal{W}}^q(X, \bullet; m) \hookrightarrow \text{Tz}^q(X, \bullet; m)$ are quasi-isomorphisms, and this shows (1).

To prove (2), recall from [32, Definition 6.2] that the differential $\delta_X : \text{Tz}^q(X, p; m) \rightarrow \text{Tz}^{q+1}(X, p+1; m)$ is defined as the push-forward with respect to the map $\delta_X : X \times \mathbb{G}_m \times \square^{n-1} \rightarrow X \times \mathbb{A}^1 \times \square^n$, induced by the map $\delta : \mathbb{G}_m \setminus \{1\} \rightarrow \mathbb{G}_m \times \square$, $\delta(x) = (x, x^{-1})$. The point of this construction is that $\delta_{X*}([Z])$ is an irreducible admissible cycle if Z is so, by [32, Lemma 6.3, Proposition 6.4]. Since the square

$$\begin{array}{ccc} X' \times \mathbb{G}_m \times \square^{n-1} & \xrightarrow{\delta_{X'}} & X' \times \mathbb{A}^1 \times \square^n \\ f \downarrow & & \downarrow f \\ X \times \mathbb{G}_m \times \square^{n-1} & \xrightarrow{\delta_X} & X \times \mathbb{A}^1 \times \square^n \end{array}$$

is transverse, it follows that $\delta_{X'^*} \circ f^*([Z]) = f^* \circ \delta_{X^*}([Z])$ for every irreducible cycle $[Z] \in \mathrm{Tz}_C^q(X, \bullet; m)$. We conclude again that $f^* \circ \delta = \delta \circ f^*$ from the quasi-isomorphism $\mathrm{Tz}_C^q(X, \bullet; m) \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)$. The proof of the theorem is complete. \square

Corollary 4.1.4. *Let $f^\sharp : R \rightarrow R'$ be a morphism of regular k -algebras essentially of finite type and let $f : \mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$ be the induced map. Then the diagram*

$$(4.2) \quad \begin{array}{ccc} \mathbb{W}_m \Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m) \\ f^* \downarrow & & \downarrow f^* \\ \mathbb{W}_m \Omega_{R'}^{n-1} & \xrightarrow{\tau_{n,m}^{R'}} & \mathrm{TCH}^n(R', n; m) \end{array}$$

commutes for all integers $m, n \geq 1$.

Proof. It is known that the left vertical arrow is a morphism of restricted Witt-complexes over R and we have seen in (4.3) that τ^R and $\tau^{R'}$ are morphisms of restricted Witt-complexes over R and R' , respectively, thus over R . By Theorem 4.1.3, the right vertical arrow is also a morphism of restricted Witt-complexes over R . In particular $f^* \circ \tau^R$ and $\tau^{R'} \circ f^*$ are both morphisms of restricted Witt-complexes over R . Since $\mathbb{W}_m \Omega_R^\bullet$ is the universal restricted Witt-complex over R , we must have $f^* \circ \tau^R = \tau^{R'} \circ f^*$. \square

4.2. The de Rham-Witt-Chow homomorphism. When k is a perfect field of characteristic $\neq 2$, it follows from [32, Theorem 1.2] that, for any regular k -algebra R essentially of finite type, the pro-system $\mathrm{TCH}^M(R) = \{\mathrm{TCH}^\bullet(R, \bullet; m)\}_{m \in \mathbb{N}}$ is a restricted Witt-complex over R , in the sense of §3.3, with respect to some operators, namely the restriction \mathfrak{R} , the differential δ , the Frobenius F_r and the Verschiebung V_r for $r \geq 1$, and the ring homomorphisms $\tau_R : \mathbb{W}_m(R) \rightarrow \mathrm{TCH}^1(R, 1; m)$. See various definitions in [32]. Hence, the universal property of the big de Rham-Witt complex $\{\mathbb{W}_m \Omega_R^\bullet\}_{m \in \mathbb{N}}$ induces the system of homomorphisms for $n, m \geq 1$

$$(4.3) \quad \tau_{n,m}^R : \mathbb{W}_m \Omega_R^{n-1} \rightarrow \mathrm{TCH}^n(R, n; m), \quad \tau_{1,m}^R = \tau_R,$$

that we would like to call the *de Rham-Witt-Chow homomorphism*. Since they give a morphism of restricted Witt-complexes over R , we deduce the following identities that we use often in this paper.

$$(4.4) \quad \tau_{n,m}^R d = \delta \tau_{n-1,m}^R; \quad \tau_{n,rm+r-1}^R V_r = V_r \tau_{n,m}^R; \quad \tau_{n,m}^R F_r = F_r \tau_{n,rm+r-1}^R.$$

The second identity implies the following variation that we use, up to applying \mathfrak{R} :

$$(4.5) \quad \tau_{n,m}^R V_r = V_r \tau_{n, \lfloor m/r \rfloor}^R.$$

When k is a field of characteristic $p > 2$, an application of the p -typicalization functor to (4.3) yields the p -typical de Rham-Witt-Chow homomorphisms

$$(4.6) \quad \tau_{n,(p)}^R : W \Omega_R^{n-1} \rightarrow \mathrm{TCH}_{(p)}^n(R; p^\infty) \quad \text{and} \quad \tau_{n,(p),i}^R : W \Omega_R^{n-1} / p^i \rightarrow \mathrm{TCH}_{(p)}^n(R; p^i).$$

We can improve [32, Theorem 1.2] for imperfect fields k as follows:

Proposition 4.2.1. *Let k be any field of characteristic $\neq 2$ and let A be a regular k -algebra essentially of finite type. Then, $\{\mathrm{TCH}^n(A, n; m)\}_{n, m \geq 1}$ has the structure of a restricted Witt-complex over A .*

Proof. We prove this theorem using Lemma 2.4.6. If k is perfect, then A is smooth over k and the proposition follows from [32, Theorem 1.2]. So suppose k is not perfect. We must then have $\mathrm{char}(k) = p > 2$. Since A is essentially of finite type over k and is regular, we can find a regular k -algebra \overline{A} of finite type and a finite set Σ of primes in \overline{A} such that $A = \overline{A}_\Sigma$, the localization at Σ .

We can find a subfield $k' \subset k$ which is finitely generated over \mathbb{F}_p , a k' -algebra \overline{R} of finite type, and a finite set Σ' of primes in \overline{R} such that $\overline{A} = \overline{R} \otimes_{k'} k$ and Σ is the extension of Σ' . Setting $A' = \overline{R}_{\Sigma'}$, we see that $A = A' \otimes_{k'} k$. Since A is a regular \mathbb{F}_p -algebra and \mathbb{F}_p is perfect, we see that A is geometrically regular over \mathbb{F}_p . But the inclusion $A' \hookrightarrow A$ of noetherian \mathbb{F}_p -algebras is faithfully flat, thus the ring A' is also geometrically regular over \mathbb{F}_p . (See [43, Lemma 10.157.3], for instance.) In particular, it is regular.

Writing k as a direct limit of its finitely generated subfields, there exists a direct system of finitely generated subfields $\{k_i\}_{i \geq 1}$ of k with $k_i \subset k_{i+1}$, and $A = \varinjlim_i A' \otimes_{k'} k_i$. We set $A_i = A' \otimes_{k'} k_i$ so that $A = \varinjlim_i A_i$. Since $A_{i+1} = A' \otimes_{k'} k_{i+1} = (A' \otimes_{k'} k_i) \otimes_{k_i} k_{i+1} = A_i \otimes_{k_i} k_{i+1}$, it follows that each $\lambda_i : A_i \rightarrow A_{i+1}$ is a faithfully flat map of noetherian \mathbb{F}_p -algebras such that $A = \varinjlim_i A_i$. Furthermore, we have $A_i \otimes_{k_i} k \simeq A' \otimes_{k'} k = A$ and hence, the inclusion $A_i \hookrightarrow A$ is also faithfully flat. We conclude as before that A_i is geometrically regular over \mathbb{F}_p and hence regular.

Since each k_i is finitely generated over \mathbb{F}_p and A_i is regular, we can find a regular \mathbb{F}_p -algebra B_i of finite type such that $A_i = S^{-1}B_i$ for some multiplicatively closed subset $S \subset B_i$. Since \mathbb{F}_p is perfect, each B_i must be smooth over \mathbb{F}_p . In particular, each A_i is a smooth \mathbb{F}_p -algebra essentially of finite type. It follows that each $\lambda_i : A_i \rightarrow A_{i+1}$ is a faithfully flat map of smooth \mathbb{F}_p -algebras essentially of finite type such that $A = \varinjlim_i A_i$. By Lemma 2.4.6, we know that the flat pull-back

$$(4.7) \quad \varinjlim_i \{\mathrm{TCH}^n(A_i, n; m)\}_{n, m \geq 1} \xrightarrow{\sim} \{\mathrm{TCH}^n(A, n; m)\}_{n, m \geq 1}$$

is an isomorphism.

Since each A_i is smooth over \mathbb{F}_p , it follows from Theorem 4.1.3 that the left hand side of (4.7) is the limit over a direct system of restricted Witt-complexes over $\{A_i\}$. It follows from [41, Proposition 1.16] (see its proof) that $\varinjlim_i \{\mathrm{TCH}^n(A_i, n; m)\}_{n, m \geq 1}$ has a natural structure of a restricted Witt-complex over A . Hence $\{\mathrm{TCH}^n(A, n; m)\}_{n, m \geq 1}$ has a structure of a restricted Witt-complex over A such that (4.7) is an isomorphism of restricted Witt-complexes over A . This finishes the proof. \square

The main theorem of this paper is the following:

Theorem 4.2.2. *Let R be a regular semi-local k -algebra essentially of finite type over any field k of characteristic $\neq 2$. Then for $m, n \geq 1$, the map $\tau_{n, m}^R : \mathbb{W}_m \Omega_R^{n-1} \rightarrow \mathrm{TCH}^n(R, n; m)$ is an isomorphism.*

This theorem generalizes the isomorphism of [41, Theorem 1], and it is the additive analogue of [10, Theorem 3.4], [24] and [25].

For such R , by [32, Theorem 7.12], we already have an isomorphism (even as rings) $\tau_{1,m}^R = \tau_R : \mathbb{W}_m(R) \xrightarrow{\sim} \mathrm{TCH}^1(R, 1; m)$, but extending it to $n \geq 2$ is very nontrivial. In this section, we first prove that $\tau_{n,m}^R$ is injective in Corollary 4.3.2. For surjectivity, we will need a somewhat technical moving lemma for additive cycles in the Milnor range in Theorem 5.2.3, and this is proved over the next several sections. This moving lemma will be used to complete the proof of the desired surjectivity in §12.

4.3. Injectivity. We suppose again that k is a perfect field of characteristic $\neq 2$ until the end of the section. As part of proof of Theorem 4.2.2, the injectivity of $\tau_{n,m}^R$ requires the following:

Proposition 4.3.1. *Let R be a regular semi-local k -algebra not necessarily essentially of finite type and let $K = \mathrm{Frac}(R)$. Then for $m \geq 1$ and $n \geq 0$, the natural map $\mathbb{W}_m \Omega_R^n \rightarrow \mathbb{W}_m \Omega_K^n$ is injective.*

Proof. Since k is perfect and R is regular, it is a filtered inductive limit of smooth semi-local essentially of finite type k -algebras by the Néron-Popescu desingularization [42]. So we can assume R is smooth essentially of finite type over k .

We prove it in two steps.

Step 1. Suppose $m = 1$, *i.e.*, we show that $\Omega_{R/\mathbb{Z}}^n \rightarrow \Omega_{K/\mathbb{Z}}^n$ is injective.

Claim: $\Omega_{R/\mathbb{Z}}^i$ is a free R -module, possibly of an infinite rank.

This is obvious for $i = 0$. Suppose $i \geq 1$. Consider the Jacobi-Zariski exact sequence of the maps $\mathbb{Z} \rightarrow k \rightarrow R$ from [35, 3.5.5.1] (which generalizes [16, Proposition 8.3A]):

$$\cdots \rightarrow D_1(R|k) \rightarrow \Omega_{k/\mathbb{Z}}^1 \otimes_k R \rightarrow \Omega_{R/\mathbb{Z}}^1 \rightarrow \Omega_{R/k}^1 \rightarrow 0,$$

where $D_1(R|k)$ is the first André-Quillen homology of M . André [2] and D. Quillen [39] (see [35, 3.5.4]). Since R is smooth over k , we have $D_1(R|k) = 0$ by [35, Theorem 3.5.6]. On the other hand, since R is a regular semi-local k -algebra, $\Omega_{R/k}^1$ is a free R -module. Thus, we have an isomorphism $\Omega_{R/\mathbb{Z}}^1 \simeq \Omega_{R/k}^1 \oplus (\Omega_{k/\mathbb{Z}}^1 \otimes_k R)$. Since $\Omega_{k/\mathbb{Z}}^1$ is a free k -module (a k -vector space), the space $\Omega_{k/\mathbb{Z}}^1 \otimes_k R$ is a free R -module. Hence, $\Omega_{R/\mathbb{Z}}^1$ is a free R -module. Taking wedge products, we deduce that $\Omega_{R/\mathbb{Z}}^i$ is a free R -module for all $i \geq 1$, proving the **Claim**.

Going back to the proof of **Step 1**, apply the functor $-\otimes_R \Omega_{R/\mathbb{Z}}^i$ to the inclusion $R \hookrightarrow K$. By **Claim**, the module $\Omega_{R/\mathbb{Z}}^i$ is free so that we get an injection $\Omega_{R/\mathbb{Z}}^i \hookrightarrow K \otimes_R \Omega_{R/\mathbb{Z}}^i$, where the latter group is isomorphic to $\Omega_{K/\mathbb{Z}}^i$ by [16, Proposition 8.2A]. Hence, **Step 1** is proven.

Step 2. Now suppose $m \geq 1$. When $\mathrm{char}(k) = 0$, by [41, Remark 1.12], $\mathbb{W}_m \Omega_R^n \rightarrow \mathbb{W}_m \Omega_K^n$ decompose into a direct product of $\Omega_{R/\mathbb{Z}}^n \rightarrow \Omega_{K/\mathbb{Z}}^n$, each of which is injective by **Step 1**. Hence, their direct product is also injective.

When $\mathrm{char}(k) = p > 0$, recall that by [18, Example 1.11] (also [41, Theorem 1.11]), $\mathbb{W}_m \Omega_R^n \rightarrow \mathbb{W}_m \Omega_K^n$ decomposes into a direct product of some copies of maps of p -typical de Rham-Witt forms $W_s \Omega_R^n \rightarrow W_s \Omega_K^n$ of finite lengths, where the product is over some values of s (see *loc.cit.* for the precise values). But, it is proven, for instance, in M. Gros [13, Proposition 5.1.2], that for any smooth k -scheme X , the Cousin complex of $W_s \Omega_X^n$ is a resolution of $W_s \Omega_X^n$. In particular,

each $W_s\Omega_R^n \rightarrow W_s\Omega_K^n$ is injective. Hence, $\mathbb{W}_m\Omega_R^n \rightarrow \mathbb{W}_m\Omega_K^n$ is injective. This completes the proof of the proposition. \square

Corollary 4.3.2. *Let R be a regular semi-local k -algebra not necessarily essentially of finite type. Then $\tau_{n,m}^R$ is injective.*

Proof. Let $K = \text{Frac}(R)$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{W}_m\Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \text{TCH}^n(R, n; m) \\ \downarrow & & \downarrow \\ \mathbb{W}_m\Omega_K^{n-1} & \xrightarrow[\simeq]{\tau_{n,m}^K} & \text{TCH}^n(K, n; m), \end{array}$$

where the right vertical map is the flat pull-back via $\text{Spec}(K) \rightarrow \text{Spec}(R)$, the map τ_K is the isomorphism of [41, Theorem 1], and the left vertical map is injective by Proposition 4.3.1. In particular, the map $\tau_{n,m}^R$ is injective. \square

4.4. A reduction. Here is one reduction on surjectivity:

Lemma 4.4.1. *If the statement of Theorem 4.2.2 holds for all regular semi-local k -algebras of geometric type, i.e., obtained by localizing at a finite set of closed points of a regular affine k -scheme, then it holds for all regular semi-local k -algebras essentially of finite type.*

Proof. Let R be any regular semi-local k -algebra essentially of finite type, i.e., $R = \mathcal{O}_{X,\Sigma}$, where X is a smooth affine k -scheme of finite type and $\Sigma \subset X$ is a finite subset. Since the map $\tau_{n,m}^R$ is injective by Corollary 4.3.2, it is enough to prove that $\tau_{n,m}^R$ is surjective.

Given any cycle class $\alpha \in \text{TCH}^n(R, n; m)$, choose a representative cycle in $\underline{\text{Tz}}^n(R, n; m)$, also denoted by α . Then, by Lemmas 2.4.1 and 2.4.3, there exists an affine open subset $U \subset X$ containing Σ such that the Zariski closure α_U of α on $U \times B_n$ is still an admissible cycle with $\partial\alpha_U = 0$. For each $p \in \Sigma$, choose a closed point $\mathfrak{m}_p \in U$ that is a specialization of p (which exists by the basic fact in commutative algebra that any proper ideal of a ring is contained in a maximal ideal). We let $\Sigma' = \{\mathfrak{m}_p \mid p \in \Sigma\}$ and let $R' := \mathcal{O}_{U,\Sigma'}$. Here $\alpha_U \in \text{TCH}^n(U, n; m)$, which gives $\alpha' \in \text{TCH}^n(R', n; m)$ by pulling back via the flat map $\text{Spec}(R') \rightarrow U$. We also have the localization map $\phi : R' \rightarrow R$, which gives the vertical flat pull-back maps in the following diagram by Corollary 4.1.4:

$$(4.8) \quad \begin{array}{ccc} \mathbb{W}_m\Omega_{R'}^{n-1} & \xrightarrow{\tau_{n,m}^{R'}} & \text{TCH}^n(R', n; m) \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ \mathbb{W}_m\Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \text{TCH}^n(R, n; m). \end{array}$$

By construction, $\phi_2(\alpha') = \alpha$. By the given assumption, Theorem 4.2.2 holds for R' so that the map $\tau_{n,m}^{R'}$ is an isomorphism. So, there is $\beta' \in \mathbb{W}_m\Omega_{R'}^{n-1}$ such that $\tau_{n,m}^{R'}(\beta') = \alpha'$. Then, by the commutativity of the diagram (4.8), we have $\tau_{n,m}^R(\phi_1(\beta')) = \alpha$. This proves the surjectivity of $\tau_{n,m}^R$ for R . \square

5. THE fs AND sfs-CYCLES

Let k be an arbitrary field. The objective of §5 ~ §11 is to prove Theorem 5.2.3, which shows that any additive higher Chow cycle in the Milnor range with trivial boundary is equivalent to a sum of very special types of cycles, called sfs-cycles. All these sections are devoted to prove this result, and unfortunately the arguments involved are rather lengthy and complicated.

We first introduce two special classes of cycles, the fs-cycles and the sfs-cycles, and address some basic properties.

Let V be a regular semi-local k -algebra essentially of finite type. Recall (Lemma 2.4.1 and Definition 2.4.2) that there is an affine atlas (X, Σ) for V (i.e., an affine k -scheme X of finite type and a finite subset $\Sigma \subset X$) such that $\text{Spec}(\mathcal{O}_{X, \Sigma}) = V$. Given an atlas (X, Σ) for V and a morphism $Z \rightarrow X$, the restriction $Z \times_X V$ will be denoted by Z_V . For any k -scheme B and a cycle $Z = \sum_{i=1}^s a_i Z_i$ on $X \times B$, where $a_i \in \mathbb{Z}$ and Z_i irreducible, we let Z_V be the cycle $\sum_{i=1}^s a_i (Z_i)_V$.

5.1. fs and sfs-cycles.

Definition 5.1.1. Let $X \in \mathbf{Sch}_k^{\text{ess}}$. We fix integers $m, n \geq 1$. Recall that for $n \geq 1$, $B_n = \mathbb{A}^1 \times \square^{n-1}$ and $\widehat{B}_n = \mathbb{P}^1 \times \overline{\square}^{n-1}$, with the coordinates (t, y_1, \dots, y_{n-1}) and we have the canonical open inclusion $B_n \subset \widehat{B}_n$ with complement F_n . For $1 \leq j \leq n$, let $\pi_j : B_n \rightarrow B_j$ and $\widehat{\pi}_j : \widehat{B}_n \rightarrow \widehat{B}_j$ be the projection maps given by $(t, y_1, \dots, y_{n-1}) \mapsto (t, y_1, \dots, y_{j-1})$. For any irreducible closed subscheme $Z \subset X \times B_n$, let $Z^{(j)} = (\text{Id}_X \times \pi_j)(Z)$. We extend it \mathbb{Z} -linearly to cycles Z on $X \times B_n$. This $Z^{(j)}$ is not necessarily a closed subscheme of $X \times B_j$ in general. However, if Z is finite over X , then the morphisms in the sequence $Z = Z^{(n)} \rightarrow Z^{(n-1)} \rightarrow \dots \rightarrow Z^{(1)} \rightarrow X$ are all finite, so each $Z^{(j)}$ is closed in $X \times B_j$.

Definition 5.1.2. Let $X = \text{Spec}(A)$ be a smooth affine k -scheme of finite type, and let Σ be a finite set of closed points on X . Let $V = \text{Spec}(\mathcal{O}_{X, \Sigma})$.

For a smooth affine geometrically integral k -variety B of dimension n , an algebraic cycle $Z \in z^n(X \times B)$ is called an *fs-cycle along Σ* , if each irreducible component Z_i of Z restricted over an affine open neighborhood U of Σ is finite and surjective over U via the projection map $X \times B \rightarrow X$. For simplicity, we often call them just fs-cycles when (X, Σ) is understood. When we need to use cycles whose components are finite and surjective over X , we will specifically say so.

In case $B = B_n$, the subgroup of admissible fs-cycles in $\text{Tz}_{\Sigma}^n(X, n; m)$ will be denoted by $\text{Tz}_{\Sigma, \text{fs}}^n(X, n; m)$.

We say that a cycle $Z \in \text{Tz}^n(X, n, m)$ is an *sfs-cycle along Σ* if for each irreducible component Z_i of Z , the following hold:

- (1) $[Z_i] \in \text{Tz}_{\Sigma, \text{fs}}^n(X, n; m)$.
- (2) For an affine open neighborhood U of Σ , $(Z_i^{(j)})_U$ is smooth over k for each $1 \leq j \leq n$.

The subgroup of sfs-cycles will be denoted by $\text{Tz}_{\Sigma, \text{sfs}}^n(X, n; m)$.

We can replace X by V and define similar groups. Namely, $\text{Tz}_{\Sigma, \text{fs}}^n(V, n; m)$ consists of cycles in $\text{Tz}_{\Sigma}^n(V, n; m)$ whose components are finite and surjective over V , and $\text{Tz}_{\Sigma, \text{sfs}}^n(V, n; m)$ consists of cycles Z in $\text{Tz}_{\Sigma, \text{fs}}^n(V, n; m)$ such that $Z_i^{(j)}$ is

smooth for each irreducible component Z_i of Z and $1 \leq j \leq n$. In this case, an sfs-cycle along Σ will be simply called an sfs-cycle. Moreover, we often suppress Σ in the notations of $\mathrm{Tz}_{\Sigma, \mathrm{fs}}^n(V, n; m)$ and $\mathrm{Tz}_{\Sigma, \mathrm{sfs}}^n(V, n; m)$.

We shall use the following notion:

Definition 5.1.3. Let $V = \mathrm{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set of closed points Σ . Let $m, n, q \geq 1$ and let $\alpha \in \mathrm{Tz}^q(V, n; m)$ be a cycle. We say that V is α -linear if there is an affine atlas (X, Σ) , where X is isomorphic to an affine space over k , such that for some $\bar{\alpha} \in \mathrm{Tz}^q(X, n; m)$, we have $\bar{\alpha}_V = \alpha$. cf. Lemma 2.4.3.

Here is one convenient result on finiteness:

Lemma 5.1.4. *Let $X \in \mathbf{SmAff}_k^{\mathrm{ess}}$ be irreducible. Let B be a smooth affine geometrically integral scheme of finite type over k of dimension $n > 0$, and let \hat{B} be a smooth projective geometrically integral scheme over k , with an open immersion $B \subset \hat{B}$.*

Let $Z \in z^n(X \times B)$ be an irreducible cycle. Then, $Z \rightarrow X$ is finite and surjective over X if and only if Z is closed in $X \times \hat{B}$.

Proof. Let $f : Z \hookrightarrow X \times \hat{B} \rightarrow X$ be the composite map. Suppose f is finite and surjective. Since the last map is projective, by [16, Corollary II-4.8-(e), Theorem II-4.9], the first map is a closed immersion. This proves (\Rightarrow) .

Conversely, suppose that Z is closed in $X \times \hat{B}$, i.e., the first map is a closed immersion (thus projective). Since the second map is projective, the composite f is projective. Hence, f is a projective morphism of affine schemes, so that it must be finite by [16, Exercise II-4.6]. Moreover, being a finite map of irreducible affine schemes of the same dimension, it must also be surjective. This proves (\Leftarrow) . \square

Lemma 5.1.5. *Let $V = \mathrm{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set of closed points Σ . Let B, \hat{B} be as in Lemma 5.1.4. Let $F := \hat{B} \setminus B$. Let $Z \in z^n(V \times B)$ be an irreducible cycle and let \hat{Z} be the Zariski closure of Z in $V \times \hat{B}$.*

Suppose that $\hat{Z} \cap (\Sigma \times F) = \emptyset$. Then given any affine atlas (X, Σ) for V , there exists an affine open subatlas (U, Σ) for V such that for the Zariski closure \bar{Z} of Z in $X \times B$, the projection map $\bar{Z}_U \rightarrow U$ is finite and surjective.

Proof. Let (X, Σ) be a given atlas. Let $\hat{\bar{Z}}$ be the Zariski closure of \bar{Z} in $X \times \hat{B}$ and let $\hat{f} : \hat{\bar{Z}} \hookrightarrow X \times \hat{B} \rightarrow X$ be the composition with the projection. Let $Y := \hat{f}(\hat{\bar{Z}} \cap (X \times F))$. Since \hat{f} is projective and since $\hat{\bar{Z}} \cap (\Sigma \times F) = \hat{Z} \cap (\Sigma \times F) = \emptyset$, we see that $Y \subset X$ is a closed subset disjoint from Σ . Hence, $X \setminus Y$ is an open neighborhood of Σ such that $\hat{\bar{Z}} \cap ((X \setminus Y) \times F) = \emptyset$. By Lemma 2.4.1, we can find an affine open neighborhood U of Σ in $X \setminus Y$, so we have $\hat{\bar{Z}} \cap (U \times F) = \emptyset$, and in particular, $\hat{\bar{Z}} \cap (U \times \hat{B}) = \bar{Z} \cap (U \times \hat{B})$. This means \bar{Z}_U is closed in $U \times \hat{B}$. Hence, by Lemma 5.1.4, the map $\bar{Z}_U \rightarrow U$ is finite and surjective. \square

We get the following characterization of fs-cycles in $\mathrm{Tz}^n(V, n; m)$:

Proposition 5.1.6. *Let $V = \operatorname{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set Σ of closed points. Let $m, n \geq 1$ and let $[Z] \in \operatorname{Tz}^n(V, n; m)$ be an irreducible cycle. Then, $[Z]$ is an fs-cycle if and only if there is an atlas (X, Σ) for V such that for the closures \overline{Z} in $X \times B_n$ and \widehat{Z} in $V \times \widehat{B}_n$, we have $[\overline{Z}] \in \operatorname{Tz}_\Sigma^n(X, n; m)$ and $\widehat{Z} \cap (\Sigma \times F_n) = \emptyset$.*

Proof. Suppose that Z is an fs-cycle. Note first that $Z \in \operatorname{Tz}_\Sigma^n(V, n; m)$. Furthermore, by Lemma 2.4.3, one can find an affine atlas (X, Σ) for V such that $\overline{Z} \in \operatorname{Tz}_\Sigma^n(X, n; m)$.

Since Z intersects $\Sigma \times B_n$ properly and $Z \cap (\Sigma \times B_n) = \overline{Z} \cap (\Sigma \times B_n)$, we have $[\overline{Z}] \in \operatorname{Tz}_\Sigma^n(X, n; m)$. Since $Z \rightarrow V$ is finite surjective, by Lemma 5.1.4, $\widehat{Z} \cap (\Sigma \times F_n) = \emptyset$.

Conversely, suppose that for an atlas (X, Σ) and the closure \overline{Z} in $X \times B_n$, we have $\overline{Z} \in \operatorname{Tz}_\Sigma^n(X, n; m)$ and $\widehat{Z} \cap (\Sigma \times F_n) = \emptyset$. Then, by Lemma 5.1.5, we may shrink (X, Σ) to an affine open atlas (U, Σ) such that $\overline{Z}_U \rightarrow U$ is finite and surjective. We still have $\overline{Z}_U \in \operatorname{Tz}_\Sigma^n(U, n; m)$ and it shows $Z \in \operatorname{Tz}_\Sigma^n(V, n; m)$. Since being finite and surjective is stable under base change to V , we deduce that $Z \rightarrow V$ is finite and surjective. Hence Z is an fs-cycle. \square

Lemma 5.1.7. *Let $V = \operatorname{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set Σ of closed points. Let $m, n \geq 1$ and let $\alpha \in \operatorname{Tz}_\Sigma^n(V, n; m)$ be a cycle each of whose components is finite over V . Then α does not intersect any proper face $F \subset \square^{n-1}$ at all.*

Proof. We can assume that $\alpha = [Z]$ is an irreducible cycle. We may further assume that Σ is a singleton set so that V is the spectrum of a regular local ring R . By taking successive quotients of R by the subsets of any chosen system of parameters for R and using an induction on the dimension of R , we can reduce to the case when R is a DVR. This induction step requires finiteness of $Z \rightarrow V$. Let $i : \{x\} \rightarrow V$ and $j : \{\eta\} \rightarrow V$ be the inclusions of the closed and the generic points of V . If Z intersects any proper face F , then either $i^*(Z)$ or $j^*(Z)$ must intersect F . On the other hand, since $\dim(Z) = \dim(V) = 1$, by dimension counting, proper intersections of Z with $\{x\} \times F$ and $\{\eta\} \times F$ imply that neither of $i^*(Z)$ or $j^*(Z)$ intersects with F . Hence Z does not intersect F . \square

Proposition 5.1.8. *Let $V = \operatorname{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set Σ of closed points. Then an irreducible cycle $[Z] \in \operatorname{Tz}^n(V, n; m)$ is an sfs-cycle if and only if there is an affine atlas $(X = \operatorname{Spec}(A), \Sigma)$ for V and an irreducible cycle $[\overline{Z}] \in \operatorname{Tz}_\Sigma^n(X, n; m)$ for which the following hold:*

- (1) $Z = \overline{Z} \times_X V$.
- (2) \overline{Z} is closed in $X \times \widehat{B}_n$, contained in $X \times \mathbb{A}^1 \times (\mathbb{A}^1)^{n-1}$, and does not intersect any proper face $F \subset \square^{n-1}$. (In particular, by Lemma 5.1.4, \overline{Z} is finite and surjective over X .)
- (3) For each $1 \leq j \leq n$, each $\overline{Z}^{(j)} = (\operatorname{Id}_X \times \pi_j)(\overline{Z}) \subset X \times B_j$ is an irreducible closed subscheme, where π_j is given by $(t, y_1, \dots, y_{n-1}) \mapsto (t, y_1, \dots, y_{j-1})$. In its coordinate ring $k[\overline{Z}^{(j)}] = A[t, y_1, \dots, y_{j-1}]/I(\overline{Z}^{(j)})$, for $a = \bar{t}$, $b_j = \bar{y}_j$ for $1 \leq j \leq n-1$, we have a sequence of finite extensions of integral

domains,

$$(5.1) \quad A \subset A[a] \subset A[a, b_1] \subset \cdots \subset A[a, b_1, \dots, b_{n-1}],$$

such that each ring in the sequence is smooth over k .

- (4) There are irreducible monic polynomials $P(t) \in A[t]$ in t and $Q_j(y_j) \in A[a, b_1, \dots, b_{j-1}][y_j]$ in y_j for $1 \leq j \leq n-1$ such that $A[a] = A[t]/(P(t))$ and $A[a, b_1, \dots, b_j] = A[a, b_1, \dots, b_{j-1}][y_j]/(Q_j(y_j))$.

Proof. For the (\Leftarrow) direction, one sees that the existence of an atlas (X, Σ) for V and a cycle $[\overline{Z}] \in \text{Tz}_\Sigma^n(X, n; m)$ satisfying (1)~(4) imply that $[Z] \in \text{Tz}_\Sigma^n(V, n; m)$ is an sfs-cycle over V . So we need to prove the converse (\Rightarrow) .

Suppose that $[Z] \in \text{Tz}^n(V, n; m)$ is an irreducible sfs-cycle. Since it is an fs-cycle by definition, by Lemma 5.1.4, Proposition 5.1.6 and Lemma 5.1.7, there is an affine atlas (X, Σ) for V such that the closure \overline{Z} of Z in $X \times B_n$ is an irreducible admissible cycle in $\text{Tz}_\Sigma^n(X, n; m)$ which satisfies (1) and (2).

Since $\overline{Z} \rightarrow X$ is finite surjective, there is a sequence of finite and surjective maps $\overline{Z} = \overline{Z}^{(n)} \rightarrow \cdots \rightarrow \overline{Z}^{(1)} \rightarrow X = \text{Spec}(A)$ such that each $\overline{Z}^{(j)}$ is finite and surjective over X . Furthermore, each $\overline{Z}_V^{(j)}$ is smooth over k because Z is an sfs-cycle.

We now want to show that there is an affine open subatlas (U, Σ) of (X, Σ) for V such that the restrictions $\overline{Z}_U^{(j)}$ to U are all smooth over k . To prove it, set $A^j = k[\overline{Z}^{(j)}]$. As each $\overline{Z}_V^{(j)}$ is smooth over k and finite over V , we see that $\Omega_{A^j/k}^1$ is a finite A -module, such that $\Omega_{S_\Sigma^{-1}A^j/k}^1$ is a free R -module, where $R = S_\Sigma^{-1}A$ for the multiplicative subset $S_\Sigma \subset A$ corresponding to the finite set of closed points Σ . Hence, there is an affine open neighborhood U of Σ in X such that each $\Omega_{A^j/k}^1|_U$ is free $k[U]$ -module. Replacing the given atlas (X, Σ) by the new (U, Σ) , we may thus assume that each $\overline{Z}^{(j)}$ is smooth over k . Hence, we proved (3).

To prove (4), we observe that if we replace A in the sequence (5.1) by its semi-local ring R by localization, then we get a sequence of finite extensions of smooth semi-local rings. Note that such rings are UFDs by Auslander-Buchsbaum and $R[a] = R[t]/I_1$ for the prime ideal $I_1 = I(Z^{(1)})$. Since $\dim(R) = \dim(R[a]) = \dim(R[t]) - 1$, we have $ht(I_1) = 1$. But, $R[t]$ is a UFD so that I_1 must be principal by [36, Proposition I.1.12A]. Thus, if $P(t) \in R[t]$ is a monic irreducible polynomial of a , then we have $I_1 = (P(t))$. Similarly, we have $R[a, b_1] = R[a][y_1]/I_2$ for the prime ideal $I_2 = I(Z^{(2)})$, and since $R[a]$ is a UFD, $R[a][y_1]$ is also a UFD. Hence, we obtain $I_2 = (Q_1(y_1))$ in the same way. Continuing this way, we get the irreducible monic polynomials $P(t) \in R[t]$ and $Q_j(y_j) \in R[a, b_1, \dots, b_{j-1}][y_j]$ for which the property (4) holds over R . Choose lifts of these polynomials over A , and then there is a localization $A' = A[f^{-1}]$ with the inclusions $A \hookrightarrow A' \hookrightarrow R$ such that the property (4) holds over A' . Replacing (X, Σ) again by $(\text{Spec}(A'), \Sigma)$, we obtain a new atlas for V for which all of (1)~(4) hold. \square

Remark 5.1.9. The reader should observe a consequence of Proposition 5.1.8 that if $[\overline{Z}] \in \text{Tz}_\Sigma^n(V, n; m)$ is an sfs-cycle, then $[Z^{(j)}] \in \text{Tz}_{\text{sfs}}^j(V, j; m)$ for $1 \leq j \leq n$.

5.2. Additive higher Chow groups of fs and sfs-cycles. We now define certain subgroups of additive higher Chow groups of smooth k -schemes in the Milnor range.

The goal is to show that these subgroups actually coincide with the additive higher Chow groups in the Milnor range for a smooth semi-local k -scheme of geometric type. This goal is achieved by means of “the fs-moving lemma” and “the sfs-moving lemma”, that we prove in the following sections.

Definition 5.2.1. Given a smooth affine k -scheme X and a finite subset of closed points Σ , we define

$$\begin{aligned}\widetilde{\mathrm{TCH}}_{\Sigma}^n(X, n; m) &= \frac{\ker(\partial : \mathrm{Tz}_{\Sigma}^n(X, n; m) \rightarrow \mathrm{Tz}^n(X, n-1; m))}{\mathrm{im}(\partial : \mathrm{Tz}^n(X, n+1; m) \rightarrow \mathrm{Tz}^n(X, n; m)) \cap \mathrm{Tz}_{\Sigma}^n(X, n; m)} \\ \mathrm{TCH}_{\Sigma, \mathrm{fs}}^n(X, n; m) &= \frac{\ker(\partial : \mathrm{Tz}_{\Sigma, \mathrm{fs}}^n(X, n; m) \rightarrow \mathrm{Tz}^n(X, n-1; m))}{\mathrm{im}(\partial : \mathrm{Tz}^n(X, n+1; m) \rightarrow \mathrm{Tz}^n(X, n; m)) \cap \mathrm{Tz}_{\Sigma, \mathrm{fs}}^n(X, n; m)} \\ \mathrm{TCH}_{\Sigma, \mathrm{sfs}}^n(X, n; m) &= \frac{\ker(\partial : \mathrm{Tz}_{\Sigma, \mathrm{sfs}}^n(X, n; m) \rightarrow \mathrm{Tz}^n(X, n-1; m))}{\mathrm{im}(\partial : \mathrm{Tz}^n(X, n+1; m) \rightarrow \mathrm{Tz}^n(X, n; m)) \cap \mathrm{Tz}_{\Sigma, \mathrm{sfs}}^n(X, n; m)}.\end{aligned}$$

Here, the definition of the group $\widetilde{\mathrm{TCH}}_{\Sigma}^n(X, n; m)$ is slightly different from that of $\mathrm{TCH}_{\Sigma}^n(X, n; m)$ in Definition 2.3.1. However, we have:

Lemma 5.2.2. *The natural surjection $\mathrm{TCH}_{\Sigma}^n(X, n; m) \rightarrow \widetilde{\mathrm{TCH}}_{\Sigma}^n(X, n; m)$ is an isomorphism.*

Proof. By the moving lemma for additive higher Chow groups of smooth affine schemes of W. Kai [22] (see [32, Theorem 4.1] for a sketch of its proof), the composition $\mathrm{TCH}_{\Sigma}^n(X, n; m) \rightarrow \widetilde{\mathrm{TCH}}_{\Sigma}^n(X, n; m) \rightarrow \mathrm{TCH}^n(X, n; m)$ is an isomorphism. Hence, the first arrow is injective. \square

If $V = \mathrm{Spec}(R)$ is a smooth semi-local k -scheme of geometric type with the set Σ of closed points, we shall write $\mathrm{TCH}_{\mathrm{fs}}^n(V, n; m)$ and $\mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m)$ for $\mathrm{TCH}_{\Sigma, \mathrm{fs}}^n(V, n; m)$ and $\mathrm{TCH}_{\Sigma, \mathrm{sfs}}^n(V, n; m)$, respectively.

Note that there are natural maps
(5.2)

$$\mathrm{TCH}_{\Sigma, \mathrm{sfs}}^n(X, n; m) \rightarrow \mathrm{TCH}_{\Sigma, \mathrm{fs}}^n(X, n; m) \rightarrow \widetilde{\mathrm{TCH}}_{\Sigma}^n(X, n; m) \rightarrow \mathrm{TCH}^n(X, n; m),$$

and the third group can be replaced by $\mathrm{TCH}_{\Sigma}^n(X, n; m)$ by Lemma 5.2.2. Our intermediate goal is to show that all arrows in this sequence are isomorphisms with X replaced by V , where V is a smooth semi-local k -scheme of geometric type:

Theorem 5.2.3 (The sfs-moving lemma). *Let k be any field. Let $r \geq 1$. Let $V = \mathrm{Spec}(R)$ be an r -dimensional smooth semi-local k -scheme of geometric type with the set Σ of closed points. Let $m, n \geq 1$ are two integers. Then the map $\mathrm{sfs}_V : \mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m) \rightarrow \mathrm{TCH}^n(V, n; m)$ is an isomorphism.*

The next result shows how we can reduce to the case when the base field is infinite perfect.

Lemma 5.2.4. *Let $V = \mathrm{Spec}(R)$ be a smooth semi-local k -scheme of geometric type with the set Σ of closed points. Suppose that the map*

$$\mathrm{sfs}_V : \mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m) \rightarrow \mathrm{TCH}^n(V, n; m)$$

is an isomorphism when k is an infinite perfect field. Then this map is an isomorphism for an arbitrary field k as well. The same holds for the map $\text{sfs}_V : \text{TCH}_{\text{fs}}^n(V, n; m) \rightarrow \text{TCH}^n(V, n; m)$.

Proof. We prove the lemma for the sfs-cycles only; the case of fs-cycles is similar. Suppose that the lemma holds when k is an infinite perfect field. First assume that k is finite (hence perfect). We use the pro- ℓ -extension trick to reduce it to the case of infinite perfect field. (see [33, Proof of Theorem 3.5.14]) Namely, choose two distinct primes ℓ_1 and ℓ_2 different from $\text{char}(k)$ and let k_i denote the pro- ℓ_i extension of k for $i = 1, 2$. Notice that these fields are infinite and perfect.

Suppose that $\alpha \in \text{TCH}_{\text{sfs}}^n(V, n; m)$ is such that $\text{sfs}_V(\alpha) = 0$. From the case of infinite perfect field, we see that $\alpha_{k_i} = 0$ for $i = 1, 2$. It follows from Proposition 5.1.8 that $\text{TCH}_{\text{sfs}}^n(V, n; m)$ commutes with the base-change of V by any direct limit of separable field extensions of k . In particular, we get two finite extensions k'_1 and k'_2 of k of relatively prime degrees such that $\alpha_{k'_i} = 0$ for $i = 1, 2$. We conclude from [27, Lemma 4.6] that $\alpha = 0$.

Next suppose that $\beta \in \text{TCH}^n(V, n; m)$. Using the case of infinite perfect field and the above property of commutativity with direct limits of separable extensions of k , we can again get two finite extensions k'_1 and k'_2 of k of relatively prime degrees such that $\beta_{k'_i} = \text{sfs}_V(\alpha_i)$ for some $\alpha_i \in \text{TCH}_{\text{sfs}}^n(V_{k'_i}, n; m)$ for $i = 1, 2$. We again conclude from [27, Lemma 4.6] that $\beta = \text{sfs}_V(\alpha)$ for some $\alpha \in \text{TCH}_{\text{sfs}}^n(V, n; m)$. This completes the case of finite fields.

Suppose now that k is an infinite imperfect field. Using Quillen's trick (see [40, Proof of Theorem 5.11, p.133]), we can find a subfield k' of k finitely generated over the prime field \mathbb{F}_p , a smooth semi-local scheme V' of geometric type over k' with the set of closed points Σ' such that $(V, \Sigma) = (V'_k, \Sigma'_k)$. Letting k_i run through the subfields of k containing k' and finitely generated over the prime field, we get $V = \varprojlim_i V'_{k_i}$.

Since each k_i is separable (see [36, Theorem 26.3]) and hence smooth over \mathbb{F}_p , it follows from Proposition 5.1.8 that $\text{TCH}_{\text{sfs}}^n(V, n; m) = \varinjlim_i \text{TCH}_{\text{sfs}}^n(V'_{k_i}, n; m)$. We also have $\text{TCH}^n(V, n; m) = \varinjlim_i \text{TCH}^n(V'_{k_i}, n; m)$ by Lemma 2.4.6. Since each V'_{k_i} is now a smooth semi-local scheme of geometric type over \mathbb{F}_p , it follows from the case of finite fields that $\text{sfs}_{V'_{k_i}}$ is an isomorphism. We conclude that $\text{sfs}_V = \varinjlim_i \text{sfs}_{V'_{k_i}}$ is an isomorphism, too. \square

6. MOVING CYCLES ON AFFINE SPACE

Our main objective for the next a few sections is to prove Theorem 5.2.3. Recall by Lemma 5.2.4, we reduce to the case when k is an infinite perfect field. So, we suppose it until the end of the proof of Theorem 5.2.3 at the end of §11.

The proof is long enough so we prove it in various steps. Here is the rough story: in Theorem 6.2.1, we first show that, if there is an admissible cycle $Z \in \text{Tz}^n(\mathbb{A}^r, n; m)$, then we can move it into a cycle in $\text{Tz}_{\Sigma, \text{sfs}}^n(U, n; m)$ for some affine open subset $U \subset \mathbb{A}^r$ containing Σ . Using this and a somewhat complicated generic projection machine, we prove in Theorem 7.4.2 that the map $\text{TCH}_{\text{fs}}^n(V, n; m) \rightarrow \text{TCH}^n(V, n; m)$ is an isomorphism. The most delicate step is to move cycles in

$\mathrm{TCH}_{\mathrm{fs}}^n(V, n; m)$ to cycles in $\mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m)$, using further generic projection machines with various loci.

The goal of §6 is to prove Theorem 6.2.1.

6.1. Spreading lemma. We consider some results that will be needed.

Lemma 6.1.1 (Spreading lemma). *Let $X = \mathrm{Spec}(R)$ be a smooth semi-local k -scheme essentially of finite type. Let K be a purely transcendental extension of k . Then the base change map*

$$p_{K/k}^* : \frac{\mathrm{Tz}^n(X, n; m)}{\mathrm{Tz}_{\mathrm{sfs}}^n(X, n; m)} \rightarrow \frac{\mathrm{Tz}^n(X_K, n; m)}{\mathrm{Tz}_{\mathrm{sfs}}^n(X_K, n; m)}$$

is injective in homology in the following sense: let $\alpha \in \mathrm{Tz}^n(X, n; m)$ be a cycle such that $\partial(\alpha) = 0$ and its base change $\alpha_K \in \mathrm{Tz}^n(X_K, n; m)$ is equivalent to a cycle in $\mathrm{Tz}_{\mathrm{sfs}}^n(X_K, n; m)$ modulo the boundary of a cycle in $\mathrm{Tz}^n(X_K, n+1; m)$. Then α is equivalent to a cycle in $\mathrm{Tz}_{\mathrm{sfs}}^n(X, n; m)$ modulo the boundary of a cycle in $\mathrm{Tz}^n(X, n+1; m)$. The same result holds if X is replaced by its atlas (X, Σ) .

Proof. Its proof is almost identical to usual spreading argument (see [27, Proposition 4.7]), except that we should check that the required smoothness, and finiteness, surjectivity over the base ring are preserved under the arguments of *loc.cit.*

By induction, we may reduce to the case when K is a purely transcendental extension of k , with $\mathrm{tr. deg}_k K = 1$, i.e., $K = k(\mathbb{A}_k^1)$. By the given assumptions, we have $\alpha_K = \beta_K + \partial\gamma_K$ for some $\beta_K \in \mathrm{Tz}_{\mathrm{sfs}}^n(X_K, n; m)$ and $\gamma_K \in \mathrm{Tz}^n(X_K, n+1; m)$. Note that each irreducible component of β_K is finite and surjective over X_K and smooth over K , and its projections to $X_K \times B_i$ for $1 \leq i \leq n-1$ are all smooth over K . Write $\beta_K = \sum_{j=1}^N r_j V_K^j$, where $r_j \in \mathbb{Z}$ and each V_K^j is an irreducible cycle in $\mathrm{Tz}_{\mathrm{sfs}}^n(X_K, n; m)$.

Then for each V_K^j , there exists a nonempty open subset $U_j \subset \mathbb{A}_k^1$ and a cycle $V_{U_j}^j \in \mathrm{Tz}^n(X \times U, n; m)$ that restricts to V_K^j , such that $V_{U_j}^j$ is finite and surjective over $X \times U_j$ and smooth over U_j , and its projections to $X \times U_j \times B_i$ for $1 \leq i \leq n-1$ are all smooth over U_j . Take $\mathcal{U} := \cap_{j=1}^N U_j$, which is yet a dense open subset. Let $V_{\mathcal{U}}^j$ be the restriction of $V_{U_j}^j$ on $X \times \mathcal{U}$. Let $\beta_{\mathcal{U}} := \sum_{j=1}^N m_j V_{\mathcal{U}}^j$.

After further shrinking \mathcal{U} if necessary, there exist cycles $\alpha_{\mathcal{U}} \in \mathrm{Tz}^n(X \times \mathcal{U}, n; m)$ and $\gamma_{\mathcal{U}} \in \mathrm{Tz}^n(X \times \mathcal{U}, n+1; m)$, that restrict to α_K and γ_K over X_K respectively, such that $\alpha_{\mathcal{U}} = \beta_{\mathcal{U}} + \partial\gamma_{\mathcal{U}}$.

Now, since k is an infinite field, the set $\mathcal{U}(k) \hookrightarrow \mathcal{U}$ is dense. In particular, we can find a point $u \in \mathcal{U}(k)$ such that the pull-backs of $\alpha_{\mathcal{U}}$, $\beta_{\mathcal{U}}$ and $\gamma_{\mathcal{U}}$ via $i_u : X \hookrightarrow X \times \mathcal{U}$ are defined. Applying i_u^* to the above equation of cycles, we get $\alpha = \iota_u^* \alpha_{\mathcal{U}} = \iota_u^* \beta_{\mathcal{U}} + \partial \iota_u^* \gamma_{\mathcal{U}}$, where we have $\iota_u^* \gamma_{\mathcal{U}} \in \mathrm{Tz}^n(X, n+1; m)$ and $\iota_u^* \beta_{\mathcal{U}} \in \mathrm{Tz}^n(X, n; m)$. But, in fact, $\iota_u^* \beta_{\mathcal{U}} \in \mathrm{Tz}_{\mathrm{sfs}}^n(X, n; m)$ because smoothness, finiteness and surjectivity of morphisms are stable under base change via ι_u . This finishes the proof of the lemma. \square

Here is a lemma that we use often:

Lemma 6.1.2 ([5, Lemma 1.2]). *Let X be an algebraic k -scheme and G a connected algebraic k -group acting on X . Let $A, B \subset X$ be closed subsets, and assume*

the fibers of the map $G \times A \rightarrow X$, $(g, a) \mapsto g \cdot a$ all have the same dimension, and that this map is dominant.

Moreover, suppose that for an overfield $K \supset k$ and a K -morphism $\psi : X_K \rightarrow G_K$, there is a nonempty open subset $U \subset X$ such that for every $x \in U_K$, a scheme point, we have $\text{tr. deg}_k k(\varphi \circ \psi(x), \pi(x)) \geq \dim G$, where $\pi : X_K \rightarrow X_k$ and $\varphi : G_K \rightarrow G_k$. Define $\phi : X_K \rightarrow X_K$ by $\phi(x) = \psi(x) \cdot x$ and suppose ϕ is an isomorphism. Then, the intersection $\phi(A_K \cap U_K) \cap B_K$ is proper.

6.2. Affine space case. The goal of §6 is to prove the following, which is weaker than a special case of Theorem 5.2.3. Later, we use it to deal with the general case.

Theorem 6.2.1. *Let $m \geq 1$. Let $\alpha \in \text{Tz}^n(\mathbb{A}_k^r, n; m)$. Let $V = \text{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \Sigma})$ for a finite subset $\Sigma \subset \mathbb{A}_k^r$ of closed points, with the localization map $j : V \rightarrow \mathbb{A}_k^r$. Then there are cycles $\beta \in \text{Tz}_{\text{sfs}}^n(V, n; m)$ and $\gamma \in \text{Tz}^n(V, n+1; m)$ such that $\partial(\gamma) = j^*(\alpha) - \beta$.*

We first discuss some lemmas. Let $\underline{x} = (x_1, \dots, x_r)$, $\underline{x}' = (x'_1, \dots, x'_r)$, and t be variables. For any $s > 0$, consider the homomorphism $\phi_s^\sharp : k[\underline{x}, t] \rightarrow k[\underline{x}, t, \underline{x}']$ of polynomial rings given by $\underline{x} \mapsto \underline{x} + t^{s(m+1)}\underline{x}'$, $t \mapsto t$. The induced map of schemes is $\phi_s : \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \mathbb{A}_k^r \rightarrow \mathbb{A}_k^r \times \mathbb{A}_k^1$, $(\underline{x}, t, \underline{x}') \mapsto (\underline{x} + t^{s(m+1)}\underline{x}', t)$. This morphism is flat, hence open. In particular, for each scheme point $g \in \mathbb{A}_k^r$, it induces a morphism $\phi_{g,s} = \phi_s(-, -, g) : \mathbb{A}_{k(g)}^r \times \mathbb{A}_{k(g)}^1 \rightarrow \mathbb{A}_{k(g)}^r \times \mathbb{A}_{k(g)}^1$, which is an isomorphism. When $g = \eta \in \mathbb{A}^r$ is the generic point, we let $K = k(\eta)$. Let $\phi_{\eta,s}^\sharp : K[\underline{x}, t] \rightarrow K[\underline{x}, t]$ be the corresponding isomorphism. In what follows, let $q_n : \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \overline{\square}_k^{n-1} \rightarrow \mathbb{A}_k^r$ be the projection to the first factor. We first prove:

Lemma 6.2.2. *Let $f(\underline{x}, t) \in k[\underline{x}, t]$ be a nonzero polynomial. Then there is a nonempty open subset $U \subset \mathbb{A}^r$, such that for each $g \in U(F)$ for some overfield $F \supset k$ and sufficiently large $s > 0$, the polynomial $\phi_{g,s}^\sharp(f)$ is monic in t , i.e., integral over $F[\underline{x}]$.*

Proof. Write $f(\underline{x}, t) = \sum_{i=0}^M f_i(\underline{x})t^{M-i}$ for some $f_i \in k[\underline{x}]$ and $M \geq 0$. Since $f \neq 0$, we have $f_0(\underline{x}) \neq 0$. Let $d_i = \deg_{\underline{x}}(f_i)$, which is the total degree in \underline{x} . We first consider the case $r = 1$. Let $c_i \in \overline{k}$ be the coefficient of the highest degree term of $f_i(\underline{x})$. Since $f_0(\underline{x}) \neq 0$, we have $c_0 \in k^\times$. Then, $f(x + t^{s(m+1)}g, t) = \sum_{i=0}^M f_i(x + t^{s(m+1)}g)t^{M-i} = \sum_{i=0}^M c_i(gt^{d_i s(m+1)} + (\text{lower degree terms in } t))t^{M-i}$. Here we suppose $g \neq 0$, i.e., $U = \mathbb{A}^1 \setminus \{0\}$. Let $F = k(g)$. Let i_0 be the smallest integer such that $d_{i_0} = \max\{d_0, d_1, \dots, d_M\}$. Here, $c_{i_0} \in k^\times$ by definition.

If $d_{i_0} = 0$, then each $f_i(x)$ is a constant, so $f(x + t^{s(m+1)}g, t)$ gives an integral relation of t as desired. Suppose $d_{i_0} > 0$. If $i_0 = 0$, then for each $i > 0$ and each $s > 0$, we have $d_0 s(m+1) + M \geq d_i s(m+1) + M > d_i s(m+1) + M - i$. Hence, the leading coefficient of the highest degree term in t is $c_0 g \in F^\times$, so, it is integral.

If $i_0 > 0$, then for each $i > i_0$ and each $s > 0$, we have $d_{i_0} s(m+1) + M - i_0 \geq d_i s(m+1) + M - i_0 > d_i s(m+1) + M - i$, while for $0 \leq i < i_0$, we have $d_i < d_{i_0}$ so that for every sufficiently large $s > 0$, we have $d_i s(m+1) + M - i < d_{i_0} s(m+1) + M - i_0$. Hence, for every sufficiently large $s > 0$, again the leading coefficient of highest degree in t is $c_{i_0} g \in F^\times$, and it gives the desired integral relation.

In case $r \geq 2$, the backbone of the proof is the same, but one problem is a possible cancellation of the highest degree terms in t , namely, if d_i is the total degree of $f_i(x_1, \dots, x_r)$, then possibly a multiple number of monomials could have the same total degree d_i . However, such g 's form a closed subscheme of \mathbb{A}^r (depends on $f(\underline{x}, t)$), so for a general $g \in U$ for some nonempty open subset $U \subset \mathbb{A}^r$, we can avoid it. \square

We first prove Theorem 6.2.1 in the special case of $n = 1$, up to a purely transcendental base change:

Lemma 6.2.3. *Let $\alpha \in \mathrm{Tz}^1(\mathbb{A}_k^r, 1; m)$, and let $V = \mathrm{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \Sigma})$ for a finite subset $\Sigma \subset \mathbb{A}_k^r$ of closed points, with the localization map $j : V \rightarrow \mathbb{A}_k^r$. Then there exist cycles $\beta \in \mathrm{Tz}_{\mathrm{sfs}}^1(V_K, 1; m)$ and $\gamma \in \mathrm{Tz}^1(V_K, 2; m)$ such that $\partial(\gamma) = j^*(\alpha_K) - \beta$.*

Proof. We may assume $\alpha = Z \subset \mathbb{A}^r \times \mathbb{A}^1$ is an integral closed subscheme. Since $\mathbb{A}^r \times \mathbb{A}^1$ is factorial, there exists an irreducible polynomial $f(\underline{x}, t) \in k[\underline{x}, t]$ such that $Z = \mathrm{Spec}(k[\underline{x}, t]/(f(\underline{x}, t)))$. The modulus condition mandates that the cycle does not intersect the divisor $\{t = 0\}$ in $\mathbb{A}^r \times \mathbb{A}^1$, so that we must have $f = th - 1$ for some $h(\underline{x}, t) \in k[\underline{x}, t]$. By Lemma 6.2.2, we can choose some $g \in \mathbb{A}^r$ and a sufficiently large $s > 0$ such that $\phi_{g,s}^\sharp(th - 1)$ is monic in t . This is equivalent to saying that $\phi_{g,s}(Z_{k(g)}) \rightarrow \mathbb{A}_{k(g)}^r$ is finite. As they are both integral and of the same dimension, this morphism is automatically surjective. In particular, by base change to $\mathrm{Spec}(K) = \mathrm{Spec}(k(\eta))$, the map $\phi_{\eta,s}(Z_K) \rightarrow \mathbb{A}_K^r$ is finite surjective.

Let $Z_{\mathrm{sm}} \subset Z$ be the set of smooth points over k . Since k is perfect, this is a dense open subset. Let $Z_{\mathrm{sing}} := Z \setminus Z_{\mathrm{sm}}$. Here, $\dim(Z_{\mathrm{sing}}) \leq r - 1$. Since $Z_{\mathrm{sm}} \rightarrow \mathrm{Spec}(k)$ is smooth, its base change $(Z_{\mathrm{sm}})_K \rightarrow \mathrm{Spec}(K)$ is also smooth.

We now apply Lemma 6.1.2 with

$$X = \mathbb{A}_k^r \times \mathbb{A}_k^1, \quad G = \mathbb{A}_k^r, \quad \psi(\underline{x}, t) = (-\eta)t^{s(m+1)}, \quad A = \Sigma \times \mathbb{A}_k^1 \quad \text{and} \quad B = Z_{\mathrm{sing}},$$

where $\eta \in \mathbb{A}_k^r$ is the generic point, G acts on $\mathbb{A}_k^r \times \mathbb{A}_k^1$ by $g \cdot (\underline{x}, t) := (g + \underline{x}, t)$. One checks that the conditions of Lemma 6.1.2 are satisfied. It follows that the intersection $\phi_{\eta,s}^{-1}(A_K) \cap B_K$ is proper. Since $\phi_{\eta,s}$ is an isomorphism, this is equivalent to saying that the intersection $A_K \cap \phi_{\eta,s}((Z_{\mathrm{sing}})_K)$ is proper. But, by dimension counting, this means $\phi_{\eta,s}((Z_{\mathrm{sing}})_K) \cap A_K = \emptyset$.

We saw $(Z_{\mathrm{sm}})_K$ is smooth over K , while its complement in Z_K is $(Z_{\mathrm{sing}})_K$. (N.B. In general, $(Z_{\mathrm{sm}})_K$ is a subset of the set of points in Z_K smooth over K .) Hence, $\phi_{\eta,s}((Z_{\mathrm{sm}})_K)$ is smooth over K , while its complement is $\phi_{\eta,s}((Z_{\mathrm{sing}})_K)$, because $\phi_{\eta,s}$ is an isomorphism. Since $\phi_{\eta,s}(Z_K) \rightarrow \mathbb{A}_K^r$ is finite as shown above, the image of $\phi_{\eta,s}((Z_{\mathrm{sing}})_K) \rightarrow \mathbb{A}_K^r$ is a proper closed subset of \mathbb{A}_K^r disjoint from Σ . Hence, there is an open neighborhood $U \subset \mathbb{A}_K^r$ of Σ , such that $\phi_{\eta,s}(Z_K) \cap q_1^{-1}(U)$ is smooth over K . Hence, $j^*(\phi_{\eta,s}(Z_K))$ is an sfs-cycle.

On the other hand, it follows from [22, Proposition 3.3, Lemma 3.5] that there is a cycle $\bar{\gamma} \in \mathrm{Tz}^n(\mathbb{A}_K^r, n + 1; m)$ such that $\partial(\bar{\gamma}) = [Z_K] - [\phi_{\eta,s}(Z_K)]$. Setting $\beta = j^*([\phi_{\eta,s}(Z_K)])$ and $\gamma = j^*(\bar{\gamma})$, we conclude $\partial(\gamma) = j^*(\alpha_K) - \beta$. \square

Before we consider the general case, we make the following observations. We consider three types of additive cycles.

Lemma 6.2.4. *Let $Z \subset \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}$ be a closed irreducible admissible subscheme of dimension r . Suppose the projection to the first factor $Z \rightarrow \mathbb{A}_k^r$ is dominant. Then there is a dense open subset $U \subset \mathbb{A}_k^r$ such that for each $g \in U$ and $s > 0$, the projection to the first factor $\phi_{g,s}(Z_{k(g)}) \rightarrow \mathbb{A}_{k(g)}^r$ is still dominant. In particular, $\dim(\phi_{g,s}(Z_{k(g)})) = \dim(Z_{k(g)})$.*

Proof. This is immediate from the definition of $\phi_{g,s}$. \square

Lemma 6.2.5. *Let $Z \subset \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}$ be a closed irreducible admissible subscheme of dimension r such that (a) the projection $q_n : Z \rightarrow \mathbb{A}_k^r$ is not dominant, while (b) the projection $pr_2 : Z \rightarrow \mathbb{A}_k^1$ is dominant. Then there is a dense open subset $U \subset \mathbb{A}_k^r$ such that for each $g \in U$ and $s > 0$, we have*

- (1) $\dim(q_n(\phi_{g,s}(Z_{k(g)}))) = \dim(q_n(Z_{k(g)})) + 1$ and
- (2) the projection $pr_2 : \phi_{g,s}(Z_{k(g)}) \rightarrow \mathbb{A}_{k(g)}^1$ is dominant.

Proof. By (b), the map pr_2 is a dominant morphism to a smooth curve, thus it is flat. In particular, $pr_2(Z) \subset \mathbb{A}_k^1$ is a dense open subset. For each $g \in \mathbb{A}_k^r$ and $s > 0$, we have a surjection $\Phi : q_n(Z_{k(g)}) \times pr_2(Z_{k(g)}) \rightarrow q_n(\phi_{g,s}(Z_{k(g)}))$, given by sending (x, t) to $x + t^{s(m+1)}g$. Thus, $\dim q_n(\phi_{g,s}(Z_{k(g)})) \leq \dim q_n(Z_{k(g)}) + 1$.

On the other hand, for each fixed closed point $t_0 \in pr_2(Z)$, the set $\Phi(q_n(Z_{k(g)}), t_0)$ has the same dimension as that of $q_n(Z_{k(g)})$, while it is a proper closed irreducible subset of $q_n(\phi_{g,s}(Z_{k(g)}))$ when g is a general member, i.e., in an open subset of \mathbb{A}_k^r . Since $pr_2(Z)$ is dense open in \mathbb{A}_k^1 and hence of positive dimension, we must have $\dim(q_n(\phi_{g,s}(Z_{k(g)}))) > \dim(q_n(Z_{k(g)}))$. The second property is obvious because $\phi_{g,s}$ does not modify the \mathbb{A}_k^1 -coordinate. \square

Lemma 6.2.6. *Let $Z \subset \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}$ be a closed irreducible admissible subscheme of dimension r such that (a) the projection to the first factor $q_n : Z \rightarrow \mathbb{A}_k^r$ is not dominant, and (b) the projection to the second factor $pr_2 : Z \rightarrow \mathbb{A}_k^1$ is not dominant, either. Let $\Sigma \subset \mathbb{A}_k^r$ be a finite subset of closed points.*

Then there is a dense open subset $U \subset \mathbb{A}_k^r$ such that for each $g \in U$, there is an open neighborhood $\mathcal{W}_g \subset \mathbb{A}_{k(g)}^r$ of Σ where $\phi_{g,s}(Z_{k(g)})$ restricted over \mathcal{W}_g is empty.

Proof. Since $pr_2 : Z \rightarrow \mathbb{A}_k^1$ is not dominant, $pr_2(Z) \subset \mathbb{A}_k^1$ is irreducible and closed. Hence, it must be a singleton $\{t_0\}$. By the modulus condition that Z satisfies, we must have $t_0 \neq 0$ and $Z \subset \mathbb{A}_k^r \times \{t_0\} \times \square_k^{n-1}$. It is therefore sufficient to prove the lemma by replacing k by $k(t_0)$ and Σ by $\pi_{t_0}^{-1}(\Sigma)$, where $\pi_{t_0} : \text{Spec}(k(t_0)) \rightarrow \text{Spec}(k)$ is the base change. We can thus assume that $t_0 \in \mathbb{A}^1(k)$. Consider the proper closed subscheme $W = \overline{q_n(Z)} \subset \mathbb{A}_k^r$ of dimension $< r$ and the dense open complement $\mathcal{U}_0 = \mathbb{A}_k^r \setminus W$.

Now for each $p \in \Sigma$, take $\mathcal{U}_p := \pi_p(p - (\mathcal{U}_0)_{k(p)})$, where $\pi_p : \mathbb{A}_{k(p)}^r \rightarrow \mathbb{A}_k^r$ is the base change. Set $\mathcal{U} = \bigcap_{p \in \Sigma} \mathcal{U}_p$. This is a finite intersection, so it is a dense open subset of \mathbb{A}_k^r . Let $U := t_0^{-s(m+1)}\mathcal{U}$, which is still dense open in \mathbb{A}_k^r . Notice that for each $g \in U$, we have $p \in \mathcal{U}_0 + t_0^{s(m+1)}g$ for each $p \in \Sigma$, so that this open set contains Σ . Let $\mathcal{W}_g := \mathcal{U}_0 + t_0^{s(m+1)}g$. Because Z restricted over \mathcal{U}_0 is empty by construction, by applying $\phi_{g,s}$ for $g \in U$, the scheme $\phi_{g,s}(Z_{k(g)})$ restricted over $\phi_{g,s}(\mathcal{U}_0) = \mathcal{W}_g$ is empty. This proves the lemma. \square

Now we prove Theorem 6.2.1 up to a purely transcendental base change:

Lemma 6.2.7. *Let $\alpha \in \text{Tz}^n(\mathbb{A}_k^r, n; m)$ and let $V = \text{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \Sigma})$ for a finite subset $\Sigma \subset \mathbb{A}_k^r$ of closed points, with the localization map $j : V \rightarrow \mathbb{A}_k^r$. Then there exist cycles $\beta \in \text{Tz}_{\text{sfs}}^n(V_K, n; m)$ and $\gamma \in \text{Tz}^n(V_K, n+1; m)$ such that $\partial(\gamma) = j^*(\alpha_K) - \beta$.*

Proof. We may assume α is irreducible, and let $Z \hookrightarrow \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}$ be the integral closed subscheme giving α . Consider the commutative diagram

$$(6.1) \quad \begin{array}{ccccc} Z & \xrightarrow{\quad} & \overline{Z} & & \\ \downarrow & & \downarrow & \searrow^{q_n} & \\ \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1} & \xrightarrow{\iota} & \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1} & \xrightarrow{p} & \mathbb{A}_k^r \times \mathbb{A}_k^1 \xrightarrow{\text{pr}} \mathbb{A}_k^r, \end{array}$$

where \overline{Z} is the Zariski closure of Z , and the maps p and pr are the obvious projections.

Case 1. First, consider the case when $q_n : Z \rightarrow \mathbb{A}_k^r$ is dominant. Let $W = p(\overline{Z}_K) \subset \mathbb{A}_K^r \times \mathbb{A}_K^1$. This is a closed subscheme of dimension r , again dominant over \mathbb{A}_K^r . This W is not necessarily admissible, but yet for the generic point $\eta \in \mathbb{A}_k^r$, $\phi_{\eta, s}(W)$ is finite and surjective over \mathbb{A}_K^r for all large $s \gg 1$ by Lemma 6.2.2. Let $\phi_{\eta, s}$ also denote the map $\phi_{\eta, s} \times \text{Id}_B : \mathbb{A}_K^r \times \mathbb{A}_K^1 \times B \rightarrow \mathbb{A}_K^r \times \mathbb{A}_K^1 \times B$ for any $B \in \mathbf{Sch}_K$. Since $\phi_{\eta, s}(\overline{Z}_K)$, $\phi_{\eta, s}(p(\overline{Z}_K))$, and \mathbb{A}_K^r are all of dimension r , and $\phi_{\eta, s}(W) \rightarrow \mathbb{A}_K^r$ is finite surjective, the map p restricted to $\phi_{\eta, s}(\overline{Z}_K) \rightarrow \phi_{\eta, s}(W)$ is quasi-finite. Since p is projective, it follows that $\phi_{\eta, s}(\overline{Z}_K)$ is finite and surjective over \mathbb{A}_K^r for $s \gg 1$.

Set $Y = \overline{Z} \setminus Z = \overline{Z} \cap (\mathbb{A}_k^r \times \mathbb{A}_k^1 \times F_n^1)$ so that $\dim_k(Y) \leq r - 1$. We now apply Lemma 6.1.2 with

$$(6.2) \quad X = \mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}; G = \mathbb{A}_k^r; \psi(\underline{x}, t, \underline{y}) = (-\eta)t^{s(m+1)}; A = \Sigma \times \mathbb{A}_k^1 \times \square_k^{n-1}; B = Y,$$

where G acts on $\mathbb{A}_k^r \times \mathbb{A}_k^1 \times \square_k^{n-1}$ by $g \cdot (\underline{x}, t, \underline{y}) = (g + \underline{x}, t, \underline{y})$. One checks immediately that the conditions of Lemma 6.1.2 are satisfied. It follows that the intersection $\phi_{\eta, s}^{-1}(A_K) \cap B_K$ is proper. By a dimension counting, this means that $\phi_{\eta, s}^{-1}(A_K) \cap B_K = \emptyset$. Equivalently, we have $A_K \cap \phi_{\eta, s}(Y_K) = \emptyset$. Since $\phi_{\eta, s}(\overline{Z}_K) \rightarrow \mathbb{A}_K^r$ is finite, we see that the image of $\phi_{\eta, s}(Y_K) \rightarrow \mathbb{A}_K^r$ is a closed subset disjoint from Σ . Taking \mathcal{W}' to be its complement, we conclude that \mathcal{W}' is an open neighborhood of Σ such that $\phi_{\eta, s}(\overline{Z}_K) \cap q_n^{-1}(\mathcal{W}'_K) = \phi_{\eta, s}(Z_K) \cap q_n^{-1}(\mathcal{W}'_K)$. Since $\phi_{\eta, s}(\overline{Z}_K) \cap q_n^{-1}(\mathcal{W}'_K) \rightarrow \mathcal{W}'_K$ is finite and surjective, we conclude that $\phi_{\eta, s}(Z_K) \cap q_n^{-1}(\mathcal{W}'_K) \rightarrow \mathcal{W}'_K$ is finite and surjective.

To show that $\phi_{\eta, s}(Z_K)$ is an sfs-cycle over some open neighborhood of Σ for all sufficiently large $s \gg 1$, we first take $A = \Sigma \times \mathbb{A}_k^1 \times \square_k^{n-1}$ and $B = \overline{Z}_{\text{sing}}$ and apply Lemma 6.1.2 with the same situation as in (6.2). We repeat the argument in the proof of Lemma 6.2.3 in verbatim to find an open neighborhood $\Sigma_K \subset U_n \subset \mathbb{A}_K^r$ such that $\phi_{\eta, s}(\overline{Z}_K) \cap q_n^{-1}(U_n)$ is smooth over K for all large $s \gg 1$. In particular, $\phi_{\eta, s}(Z_K) \cap q_n^{-1}(U_n)$ is smooth over K for all large $s \gg 1$.

Since $\phi_{\eta, s}(\overline{Z}_K)$ finite over \mathbb{A}_K^r , we see that each projection of \overline{Z}_K , via $\pi_j : \mathbb{A}_K^r \times \mathbb{A}_K^1 \times \square_K^{n-1} \rightarrow \mathbb{A}_K^r \times \mathbb{A}_K^1 \times \square_K^{j-1}$, is finite over \mathbb{A}_K^r for $1 \leq j \leq n$. We can now

apply the above argument to each of these projections successively to get open neighborhoods $\Sigma_K \subset U_j \subset \mathbb{A}_K^r$ such that

- (1) $\phi_{\eta,s}(\overline{Z}_K) \cap q_n^{-1}(U_j) = \phi_{\eta,s}(Z_K) \cap q_n^{-1}(U_j)$,
- (2) $\phi_{\eta,s}(\overline{Z}_K) \cap q_n^{-1}(U_j)$ is finite and surjective over U_j , and
- (3) $\pi_i(\phi_{\eta,s}(\overline{Z}_K) \cap q_n^{-1}(U_j))$ is smooth over K for all $j \leq i \leq n$.

Setting $\mathcal{W} = \mathcal{W}_K' \cap (\bigcap_{j=1}^n U_j)$ and noting that $\phi_{\eta,s}$ commutes with each projection $\mathbb{A}_K^r \times \mathbb{A}_K^1 \times \overline{\square}_K^j \rightarrow \mathbb{A}_K^r \times \mathbb{A}_K^1 \times \overline{\square}_K^{j-1}$, we conclude that $\mathcal{W} \subset \mathbb{A}_K^r$ is an open neighborhood of Σ_K such that $\phi_{\eta,s}(Z_K) \cap q_n^{-1}(\mathcal{W})$ is an sfs-cycle over \mathcal{W} . Since each $\phi_{g,s}(Z)$ is congruent to Z as an admissible additive cycle by [22, Proposition 3.3, Lemma 3.5], we are done in this case.

Case 2. We next consider the case when the map $Z \rightarrow \mathbb{A}_k^r$ is not dominant. Here, we have two further cases.

Subcase 2-1. Suppose first that the projection $pr_2 : Z \rightarrow \mathbb{A}_k^1$ is dominant. In this case, we can apply Lemma 6.2.5 repeatedly to conclude that the composite $\phi_{g_1,s} \circ \cdots \circ \phi_{g_c,s}(Z) \rightarrow \mathbb{A}_k^r$ is dominant for a finite sequence of general members g_1, \dots, g_c of \mathbb{A}_k^r . So, *Subcase 2-1* reduces to the case when $Z \rightarrow \mathbb{A}_k^r$ is dominant, which we already treated in Case 1.

Subcase 2-2. The only case remaining is when neither of the maps $Z \rightarrow \mathbb{A}_k^r$ and $Z \rightarrow \mathbb{A}_k^1$ is dominant. In this case, we see by Lemma 6.2.6 that there is an open subset $\mathcal{W} \subset \mathbb{A}_k^r$ containing Σ such that the map $\phi_{\eta,s}(Z_K) \cap q_n^{-1}(\mathcal{W}_K) = \emptyset$. In particular, $[j^*(\phi_{\eta,s}(Z_K))] = 0$, where $j : V \hookrightarrow \mathbb{A}_k^r$ is the localization map. So, by applying [22, Proposition 3.3, Lemma 3.5], we conclude that such Z_K is equivalent to 0. The proof of the lemma is now complete. \square

Proof of Theorem 6.2.1. This follows immediately from Lemmas 6.1.1 and 6.2.7. \square

7. THE FS-MOVING LEMMA

We continue to suppose that k is an infinite perfect field, unless stated otherwise. As an intermediate step, the goal of this section is to prove Theorem 7.4.2, which says that the map $\text{fs}_V : \text{TCH}_{\text{fs}}^n(V, n; m) \rightarrow \text{TCH}^n(V, n; m)$ is an isomorphism for a regular semi-local k -scheme V of geometric type. In order to answer this question, we need various techniques of linear projections inside projective or affine spaces.

7.1. Some algebraic results. We collect some algebraic results before we delve into linear projections.

Lemma 7.1.1. *Let $A \xrightarrow{g} B \xrightarrow{f} C$ be local flat morphisms of noetherian local rings. Assume that fg and f are étale. Then g is also étale.*

Proof. We only need to show that $\Omega_{B/A}^1 = 0$. However, as f is étale, the relative differential and the André-Quillen homology of C over B vanish. This implies in particular that $0 = \Omega_{C/A}^1 \simeq \Omega_{B/A}^1 \otimes_B C$. Since f is faithfully flat, we deduce that $\Omega_{B/A}^1 = 0$. \square

Lemma 7.1.2. *Let $f : A \rightarrow B$ be an injective finite unramified local morphism of noetherian local rings which induces an isomorphism of the residue fields. Then f is an isomorphism.*

Proof. Let \mathfrak{m}_A and \mathfrak{m}_B denote the maximal ideals of A and B , respectively. We need to show that f is surjective. Using the finiteness of f and Nakayama's lemma, it suffices to show that $A/\mathfrak{m}_A \rightarrow B/\mathfrak{m}_A$ is surjective. But this follows because the map $A/\mathfrak{m}_A \rightarrow B/\mathfrak{m}_B$ is an isomorphism and so is the map $B/\mathfrak{m}_A \rightarrow B/\mathfrak{m}_B$ as f is unramified. \square

Lemma 7.1.3. *Let $f : X \rightarrow Y$ be a finite and flat map of connected smooth k -schemes. Let $W \subset Y$ be an irreducible closed subset and let $y \in W$ be a closed point. Set $S = f^{-1}(y)$ and $W' = X \times_Y W$. Let $x \in f^{-1}(y)$ and let $Z \subset W'$ be an irreducible component with $x \in Z$. Assume that f is étale at x and $k(y) \xrightarrow{\sim} k(x)$. Then, $Z \cap S = \{x\}$ if and only if Z is the only component of W' passing through x .*

Proof. First, suppose $S = \{x\}$. We claim that f is an isomorphism locally around y , so that the lemma holds trivially. Indeed, it follows from Lemma 7.1.2 that the map $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is an isomorphism. This in turn implies that f is a finite and flat map with $[k(X) : k(Y)] = 1$ (see [34, Ex. 5.1.25]) and hence must be an isomorphism.

We now suppose $|S| > 1$. Consider the commutative diagram of semi-local rings:

$$(7.1) \quad \begin{array}{ccccc} \mathcal{O}_{Y,y} & \xrightarrow{\alpha_1} & \mathcal{O}_{X,S} & \xrightarrow{\alpha_2} & \mathcal{O}_{X,x} \\ \downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta_3 \\ \mathcal{O}_{W,y} & \xrightarrow{\alpha_3} & \mathcal{O}_{W',S} & \xrightarrow{\alpha_4} & \mathcal{O}_{W',x} \\ & \searrow \gamma & \downarrow \beta_4 & & \downarrow \beta_5 \\ & & \mathcal{O}_{Z,S} & \xrightarrow{\alpha_5} & \mathcal{O}_{Z,x} \end{array}$$

γ' (curved arrow from $\mathcal{O}_{W,y}$ to $\mathcal{O}_{Z,x}$)

Here, α_1 and α_3 are finite and flat maps, and $\alpha_2 \circ \alpha_1$ is étale. Using this, we see that the lemma is equivalent to that α_5 is an isomorphism if and only if β_5 is so.

Assume first that α_5 is an isomorphism. Since β_4 is surjective and α_3 is finite, we conclude that γ is finite. Thus, γ' is a finite map of local rings.

Next, since $\alpha_2 \circ \alpha_1$ is étale, we see that $\alpha_4 \circ \alpha_3$ is also étale. Since β_5 is surjective, we see that γ' is unramified. Thus, γ' is a finite and unramified map of local rings. Since $Z \rightarrow W$ is surjective and $k(y) \simeq k(x)$, by Lemma 7.1.2, γ' is an isomorphism. In particular, $\alpha_4 \circ \alpha_3$ is an étale map of local rings such that $\beta_5 \circ \alpha_4 \circ \alpha_3$ is an isomorphism, in particular, étale. It follows easily that β_5 is étale. Thus, β_5 is a surjective étale map of local rings, but it can happen only if β_5 is an isomorphism.

Conversely, suppose that β_5 is an isomorphism. Let \mathfrak{p} denote the minimal prime of $\mathcal{O}_{W',S}$ such that $\mathcal{O}_{W',S}/\mathfrak{p} = \mathcal{O}_{Z,S}$ and let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ denote the set of distinct minimal primes of $\mathcal{O}_{W',S}$ different from \mathfrak{p} . To show that α_5 is an isomorphism, we need to show that $\mathfrak{p} + \mathfrak{p}_i = \mathcal{O}_{W',S}$ for all $1 \leq i \leq m$.

We first make the following

Claim 1: $\mathfrak{p}_i \mathcal{O}_{W',x} = \mathcal{O}_{W',x}$ for all $1 \leq i \leq m$.

(\because) Note that $\mathcal{O}_{W',x}$ is an integral domain because $\mathcal{O}_{Z,x}$ is an integral domain and β_5 is an isomorphism. Thus, we must have either $\mathfrak{p}_i \mathcal{O}_{W',x} = 0$, or $\mathfrak{p}_i \mathcal{O}_{W',x} = \mathcal{O}_{W',x}$. In the first case, we have $\mathfrak{p}_i \mathcal{O}_{Z,x} = 0$ as β_5 is an isomorphism. Equivalently, $\alpha_5 \circ \beta_4(\mathfrak{p}_i) = 0$.

Since $\mathfrak{p}_i \neq \mathfrak{p}$, and $\mathfrak{p}_i, \mathfrak{p}$ are minimal, there is $a_i \in \mathfrak{p}_i \setminus \mathfrak{p}$ such that $\beta_4(a_i) \neq 0$. Hence, $\alpha_5 \circ \beta_4(a_i) \neq 0$, because α_5 is injective being a localization of an integral domain. This is a contradiction. Thus, we must have $\mathfrak{p}_i \mathcal{O}_{W',x} = \mathcal{O}_{W',x}$ for each i , proving Claim 1.

Let \mathfrak{m} be the maximal ideal of $\mathcal{O}_{W',S}$ defining the point x . Using Claim 1, we see that for any given $1 \leq i \leq m$, there exists $a_i \in \mathfrak{p}_i \setminus \mathfrak{m}$ in $\mathcal{O}_{W',S}$ such that $\alpha_4(a_i)$ is invertible. Let $a = \prod_{i=1}^m a_i$. We see that there are nonzero elements $b, c \in \mathcal{O}_{Z',S}$ with $c \notin \mathfrak{m}$ such that $c(1 - ab) = 0$.

Claim 2: $1 - ab \in \mathfrak{p}$.

(\because) Let $v = 1 - ab$. Then, we have $cv = 0 \in \mathfrak{m}$ with $c \notin \mathfrak{m}$, so that $v \in \mathfrak{m}$ and $\alpha_4(v) = 0$. If $v \notin \mathfrak{p}$, then $v \in \mathfrak{m} \setminus \mathfrak{p}$, so that $\beta_4(v) \neq 0$. This in turn means that $\beta_5 \circ \alpha_4(v) = \alpha_5 \circ \beta_4(v) \neq 0$. This contradicts that $\alpha_4(v) = 0$. Hence, we have $v \in \mathfrak{p}$, proving Claim 2.

Using Claim 2, we see that $v \in \mathfrak{p}$, $ab \in \mathfrak{p}_i$ for all i , while $v - ab = 1$. This shows that $\mathfrak{p} + \mathfrak{p}_i = \mathcal{O}_{W',S}$ for all $1 \leq i \leq m$, thus α_5 is an isomorphism. \square

7.2. Linear projections. We recall some known facts and set up our terminologies used throughout our proofs of the fs-moving and the sfs-moving lemmas in the next several sections.

Notations: Given integers $0 \leq n < N$, let $\text{Gr}(n, \mathbb{P}_k^N)$ denote the Grassmannian scheme of n -dimensional linear subspaces of \mathbb{P}_k^N . A point $[x_0, \dots, x_n] \in \mathbb{P}_k^n$ is often denoted in short by $[x]$.

Given two closed subschemes $Y, Y' \subset \mathbb{P}_k^N$, let $\text{Sec}(Y, Y')$ denote the union of all lines $\ell_{yy'}$ joining distinct points $y \in Y, y' \in Y'$. This is called the *join* of Y and Y' . If these are linear subspaces of \mathbb{P}_k^N , one checks that $\dim(\text{Sec}(Y, Y')) = \dim(Y) + \dim(Y') - \dim(Y \cap Y')$ with the convention that $\dim(\emptyset) = -1$. In general, we have $\dim(\text{Sec}(Y, Y')) \leq \dim(Y) + \dim(Y') + 1$. If $Y = Y'$, the scheme $\text{Sec}(Y, Y') = \text{Sec}(Y)$ is the secant variety of Y . If $Y' = L$ is a linear subspace, then $\text{Sec}(Y, L) = C_L(Y)$ is the cone over Y with vertices in L .

Given a locally closed subset $S \subset \mathbb{P}_k^N$, we denote the set of n -dimensional linear subspaces of \mathbb{P}_k^N which *do not* intersect S by $\text{Gr}(S, n, \mathbb{P}_k^N)$. If $S = \{x\}$, we write $\text{Gr}(S, n, \mathbb{P}_k^N)$ as $\text{Gr}(x, n, \mathbb{P}_k^N)$.

We denote the set of n -dimensional linear subspaces of \mathbb{P}_k^N *containing* a locally closed subscheme $S \subset \mathbb{P}_k^N$ by $\text{Gr}_S(n, \mathbb{P}_k^N)$. We write $\text{Gr}_S(n, \mathbb{P}_k^N)$ as $\text{Gr}_x(n, \mathbb{P}_k^N)$ if $S = \{x\}$ is a closed point. One checks that $\text{Gr}(n, \mathbb{P}_k^N)$ is a homogeneous space of dimension $(N-n)(n+1)$. If $M \subset \mathbb{P}_k^N$ is a linear subspace of dimension $0 \leq m \leq n$, then $\text{Gr}_M(n, \mathbb{P}_k^N)$ is a homogeneous space which is an irreducible closed subscheme of $\text{Gr}(n, \mathbb{P}_k^N)$ of dimension $(N-n)(n-m)$. Given two distinct locally closed subsets $S, S' \subset \mathbb{P}_k^N$, we let $\text{Gr}_S(S', n, \mathbb{P}_k^N) := \text{Gr}_S(n, \mathbb{P}_k^N) \cap \text{Gr}(S', n, \mathbb{P}_k^N)$.

For a scheme X , let $X_{\text{sing}} \subset X$ be the singular locus of X and let X_{sm} be its complement. For a closed subscheme $X \subset \mathbb{P}_k^N$, let $\text{Gr}^{\text{tr}}(X, n, \mathbb{P}_k^N)$ denote the set of n -dimensional linear subspaces which *do not* intersect X_{sing} , and possibly intersect X_{sm} transversely. We let $\text{Gr}^{\text{tr}}(S, X, n; \mathbb{P}_k^N) = \text{Gr}(S, n; \mathbb{P}_k^N) \cap \text{Gr}^{\text{tr}}(X, n; \mathbb{P}_k^N)$.

In the above, for any linear subscheme $L \subset \mathbb{P}_k^N$, we can also define $\text{Gr}(S, n, L)$, $\text{Gr}_S(n, L)$, or $\text{Gr}^{\text{tr}}(X, n, L)$, similarly. The following result is elementary.

Lemma 7.2.1. *For any finite set $S \subset \mathbb{P}_k^N$, $\text{Gr}(S, n, \mathbb{P}_k^N) \subset \text{Gr}(n, \mathbb{P}_k^N)$ is a dense open subset. Moreover, if $S' \subset S$, then $\text{Gr}(S, n, \mathbb{P}_k^N) \subset \text{Gr}(S', n, \mathbb{P}_k^N)$.*

Proof. The lemma is easily reduced to showing that $\text{Gr}(x', n, \mathbb{P}_k^N) \cap \text{Gr}_x(n, \mathbb{P}_k^N) \neq \emptyset$ whenever $x \neq x' \in \mathbb{P}_k^N$. But this is immediate since $N > n$. \square

7.2.1. Affine Veronese embedding. Recall that for positive integers $m, d \geq 1$, the Veronese embedding $v_{m,d} : \mathbb{P}_k^m \hookrightarrow \mathbb{P}_k^N$ is a closed embedding given by $v_{m,d}([x]) = [M_0(x), \dots, M_N(x)] = [M(\underline{x})]$, where $N = \binom{m+d}{d} - 1$ and $\{M_0, \dots, M_N\}$ are all monomials in $\{x_0, \dots, x_m\}$ of degree d , arranged in the lexicographic order.

If $[y_0, \dots, y_N] \in \mathbb{P}_k^N$ are the projective coordinates, it is clear that $v_{m,d}^{-1}(\{y_0 = 0\}) = \{x_0^d = 0\}$. In particular, the Veronese embedding yields Cartesian squares

$$(7.2) \quad \begin{array}{ccccc} \mathbb{A}_k^m & \rightarrow & \mathbb{P}_k^m & \leftarrow & dH_{m,0} \\ \downarrow v_{m,d} & & \downarrow v_{m,d} & & \downarrow v_{m,d} \\ \mathbb{A}_k^N & \rightarrow & \mathbb{P}_k^N & \leftarrow & H_{N,0}, \end{array}$$

where $H_{m,0} \subset \mathbb{P}_k^m$ is the hyperplane $\{x_0 = 0\}$ and the vertical arrows are all closed embeddings. The closed embedding $v_{m,d} : \mathbb{A}_k^m \hookrightarrow \mathbb{A}_k^N$ is given by

$$(7.3) \quad v_{m,d}(y_1, \dots, y_m) = (M_1, \dots, M_N),$$

where $\{M_1, \dots, M_N\}$ is the ordered set of all monomials in $\{y_1, \dots, y_m\}$ of degree bounded by d . It will be called the *affine Veronese embedding* in the sequel.

7.2.2. Linear projections. Let's say two linear subspaces $L, L' \subset \mathbb{P}_k^N$ are *complementary* if $L \cap L' = \emptyset$ and $\text{Sec}(L, L') = \mathbb{P}_k^N$. Given two complementary linear subspaces L and L' of dimensions $N - r - 1$ and r , respectively, there is a linear projection morphism $\phi_L : \mathbb{P}_k^N \setminus L \rightarrow L'$. Up to a linear change of coordinates in \mathbb{P}_k^N , this map can be defined by the sections $\{s_0, \dots, s_r\}$ of $\mathcal{O}_{\mathbb{P}_k^N}(1)$ such that $\{s_0, \dots, s_r, s_{r+1}, \dots, s_N\}$ is a basis of $H^0(\mathbb{P}_k^N, \mathcal{O}(1))$. Notice that ϕ_L defines a vector bundle morphism over L' of rank $N - r$, whose fiber over a point $x \in L'$ is the affine space $C_x(L) \setminus L$, where $C_x(L) = \text{Sec}(x, L)$.

Observe that if $H \subset \mathbb{P}_k^N$ is a hyperplane containing L' and $X \subset \mathbb{P}_k^N$ is a closed subscheme with $X \cap L = \emptyset$ (in particular, $\dim(X) \leq r$), then ϕ_L defines the Cartesian squares of morphisms

$$(7.4) \quad \begin{array}{ccccc} X \setminus H & \rightarrow & X & \leftarrow & X \cap H \\ \downarrow & & \downarrow & & \downarrow \\ L' \setminus H & \rightarrow & L' & \leftarrow & L' \cap H. \end{array}$$

Since $X \rightarrow L'$ is projective with affine fibers, it must be finite. In particular, the map $X \setminus H \rightarrow L' \setminus H$ of affine schemes is also finite. We identify $L' \simeq \mathbb{P}_k^r$. Together with (7.2), we deduce the following fact, which we use often:

Lemma 7.2.2. *Let $X \hookrightarrow \mathbb{A}_k^m$ be an affine scheme of dimension $r \geq 1$ and let $\overline{X} \hookrightarrow \mathbb{P}_k^m$ be its projective closure. Then, for $d \geq 1$, the Veronese embedding*

$v_{m,d} : \mathbb{P}_k^m \hookrightarrow \mathbb{P}_k^N$ and the linear projection away from $L \in \text{Gr}(N - r - 1, \mathbb{P}_k^N)$ yield a Cartesian diagram with finite vertical maps:

$$(7.5) \quad \begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \phi_L \downarrow & & \downarrow \phi_L \\ \mathbb{A}_k^r & \longrightarrow & \mathbb{P}_k^r, \end{array}$$

if $L \in \text{Gr}(X, N - r - 1, H_{N,0}) \subset \text{Gr}(N - r - 1, \mathbb{P}_k^N)$, where $H_{N,0} = \{y_0 = 0\} \subset \mathbb{P}_k^N$ as in (7.2).

To ensure that the collection of such linear subspaces is nonempty, we use the following often.

Lemma 7.2.3. *Assume that k is algebraically closed. Let $X \subset \mathbb{P}_k^N$ be a closed subscheme of dimension $r \geq 1$ with $N \gg r$ and let $H \hookrightarrow \mathbb{P}_k^N$ be a hyperplane, not containing X . Then $\text{Gr}(X, N - r - 1, H)$ is a dense open subset of $\text{Gr}(N - r - 1, H)$.*

Proof. Consider the incidence variety $S = \{(x, L) \in X \times \text{Gr}(N - r - 1, H) \mid x \in L\}$ and let $Y := X \cap H$. We have the obvious projection maps

$$(7.6) \quad X \xleftarrow{\pi_1} S \xrightarrow{\pi_2} \text{Gr}(N - r - 1, H).$$

The fiber of π_1 over $X \setminus Y$ is empty and it is a smooth fibration over Y with each fiber isomorphic to $\text{Gr}(N - r - 2, \mathbb{P}_k^{N-2})$. It follows that $\dim(S) = \dim(Y) + \dim \text{Gr}(N - r - 2, \mathbb{P}_k^{N-2}) = r - 1 + r(N - r - 1) = r(N - r) - 1$. Thus, $\pi_2(S)$ is a closed subscheme of $\text{Gr}(N - r - 1, H)$ of dimension $\leq r(N - r) - 1$ which is less than $\dim \text{Gr}(N - r - 1, H) = r(N - r)$. Hence, $\text{Gr}(X, N - r - 1, H) = \text{Gr}(N - r - 1, H) \setminus \pi_2(S)$ is a dense open subset. \square

Lemma 7.2.4. *Assume that k is algebraically closed. Let $r \geq 2$ be an integer and assume $N \gg r$. Let $H \hookrightarrow \mathbb{P}_k^N$ be a hyperplane. Let $L \subset \mathbb{P}_k^N$ be a linear subspace of dimension $N - r + 1$ intersecting H transversely and let $X \subset L$ be a curve (not necessarily connected). Then the set of linear subspaces in $\text{Gr}^{\text{tr}}(L, N - 2, H)$ which do not intersect X , is a dense open subset of $\text{Gr}(N - 2, H)$.*

Proof. Observe that $\text{Gr}^{\text{tr}}(L, N - 2, H)$ is a dense open subset of $\text{Gr}(N - 2, H)$. Consider the map $\nu_L : \text{Gr}^{\text{tr}}(L, N - 2, H) \rightarrow \text{Gr}(N - r - 1, L \cap H)$ given by $\nu_L(M) = L \cap M$. This ν_L is a smooth surjective morphism of relative dimension $2(r - 1)$. It follows from Lemma 7.2.3 that $\text{Gr}(X, N - r - 1, L \cap H)$ is a dense open subset of $\text{Gr}(N - r - 1, L \cap H)$, so $\nu_L^{-1}(\text{Gr}(X, N - r - 1, L \cap H))$ is a dense open subset of $\text{Gr}^{\text{tr}}(L, N - 2, H)$, and hence a dense open subset of $\text{Gr}(N - 2, H)$. \square

7.3. The fs-moving lemma. Let X be a smooth affine k -scheme of dimension $r \geq 1$. Let B be a geometrically integral smooth affine k -scheme of positive dimension with a geometrically integral smooth projective compactification \widehat{B} such that $F := \widehat{B} \setminus B$ is an effective divisor.

Notations : Let \overline{X} be a compactification of X and let $W \subset \overline{X} \times \widehat{B}$ be a closed subscheme. Let $\widehat{f} : W \rightarrow \overline{X}$ and $\widehat{g} : W \rightarrow \widehat{B}$ be the composites with the

projections. Then for any morphisms $X' \rightarrow \overline{X}$ and $B' \rightarrow \widehat{B}$, we denote by $W_{X'}$ and $W^{B'}$ the schemes defined by the Cartesian squares

$$(7.7) \quad \begin{array}{ccc} W_{X'} & \rightarrow & W \\ \downarrow & & \downarrow \widehat{f} \\ X' & \rightarrow & \overline{X} \end{array} \quad \begin{array}{ccc} W^{B'} & \rightarrow & W \\ \downarrow & & \downarrow \widehat{g} \\ B' & \rightarrow & \widehat{B}. \end{array}$$

In particular, for any $x \in \overline{X}$, we write $\widehat{f}^{-1}(x)$ as W_x .

Definition 7.3.1. Choose a closed embedding $X \hookrightarrow \mathbb{A}_k^m$ and let \overline{X} be its closure in \mathbb{P}_k^m . For an $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$, let $\phi_L : \overline{X} \rightarrow \mathbb{P}_k^r$ be the finite projection map as in §7.2.2. Let $\phi_{L,A} : \overline{X} \times A \rightarrow \mathbb{P}_k^r \times A$ be the map $\phi_L \times \text{Id}_A$ for any k -scheme A . We often write it just as ϕ_L .

Given a finite and flat map $h : X \rightarrow X'$ and a cycle Z on $X \times B$, the cycle $h^* \circ h_*([Z]) - [Z]$ is called the *residual cycle* of $[Z]$, and denoted by $h^+([Z])$. In case $Z \in \text{Tz}^q(X, n; m)$, the flat pull-back h^* and the projective push-forward h_* of additive cycles exist (see [27, Theorem 3.1], [26, §3.3, 3.4]).

The reader should observe that $[Z]$ may still be a component of $h^+([Z])$ in general. By extending linearly, $h^+(-)$ is defined on the full cycle complex $\text{Tz}^q(X, n; m)$. For a map $\phi_L : X \rightarrow \mathbb{A}_k^r$ as above, we write $\phi_L^+([Z])$ as $L^+([Z])$.

Let $Z \subset X \times B$ be a closed subscheme of dimension r and let $\widehat{Z} \subset X \times \widehat{B}$ be its Zariski closure. (Later, we will consider $B = \mathbb{A}_k^1 \times \square_k^{n-1}$ and $\widehat{B} = \widehat{B}_n = \mathbb{P}_k^1 \times \square_k^{n-1}$.)

Let $f : Z \rightarrow X$ and $\widehat{f} : \widehat{Z} \rightarrow X$ be composites with the projections. Let $\{Z_1, \dots, Z_s\}$ be the set of irreducible components of Z . For a given $1 \leq i \leq s$, we fix closed points $x_i \in X$, $b_i \in B$ such that $\alpha_i = (x_i, b_i) \in Z_i$, with $\alpha_i \notin Z_j$ if $j \neq i$. Since each $Z_i \neq \emptyset$, such closed points always exist. Let $D_0 \subset X$ be a finite set of closed points containing $\{x_1, \dots, x_s\}$ and let $E^0 \subset B$ be a finite set of closed points containing $\{b_1, \dots, b_s\}$. Set $F^0 = F \cup E^0$.

For every $x \in \overline{X}$, let $L^+(x) = \phi_L^{-1}(\phi_L(x)) \setminus \{x\}$ and let $L^+(D_0) = \phi_L^{-1}(\phi_L(D_0)) \setminus D_0$. Under these conditions, we wish to prove the following.

Lemma 7.3.2. *Assume that X is not isomorphic to an affine space over k and that no component of Z is contained in $X \times E^0$. After replacing the embedding $\overline{X} \hookrightarrow \mathbb{P}_k^m$ by its composition with a suitable Veronese embedding $\mathbb{P}_k^m \hookrightarrow \mathbb{P}_k^N$, a general $L \in \text{Gr}(\overline{X}, N - r - 1, H_{N,0})$ satisfies the following, where $H_{N,0}$ is as in (7.2).*

- (1) ϕ_L is étale at D_0 .
- (2) $\phi_L(x) \neq \phi_L(x')$ for $x \neq x' \in D_0$.
- (3) $k(\phi_L(x)) \xrightarrow{\sim} k(x)$ for all $x \in D_0$.
- (4) $L^+(x) \neq \emptyset$ for all $x \in D_0$.
- (5) $L^+(D_0) \cap \widehat{f}(\widehat{Z}^{F^0}) = \emptyset$.

For any $1 \leq i \leq s$ such that $f : Z_i \rightarrow X$ is not dominant, we also have

- (6) $L^+(D_0) \cap \widehat{f}(\widehat{Z}_i) = \emptyset$.

Proof. Replacing the given embedding $\overline{X} \hookrightarrow \mathbb{P}_k^m$ by its composition with a Veronese embedding, we may begin with a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ such that $N \gg r$

and the degree of \overline{X} in \mathbb{P}_k^N is bigger than one. Let $\overline{X}_{\text{sing}}$ denote the singular locus of \overline{X} . Since k is perfect, we see that $\dim(\overline{X}_{\text{sing}}) \leq r - 1$.

Step 1. *We first assume that k is algebraically closed.* Let $W \subset \overline{X} \times \widehat{B}$ be the closure of \widehat{Z} in $\overline{X} \times \widehat{B}$. Since Z is an r -dimensional scheme none of whose component is contained in $X \times E^0$, and $F \subset \widehat{B}$ is an effective divisor, we see that $\dim(W^{F^0}) \leq r - 1$. Since \widehat{f} is projective, we see that $\widehat{f}(W^{F^0})$ is a closed subscheme of \overline{X} of dimension $\leq r - 1$.

Let Z_{nd} be the union of irreducible components of Z which are not dominant over X and let \widehat{Z}_{nd} be its closure in $\overline{X} \times \widehat{B}$. Then, $\widehat{f}(\widehat{Z}_{\text{nd}})$ is a closed subscheme of \overline{X} of dimension $\leq r - 1$. So, $\mathcal{X}^{F^0} := \overline{X}_{\text{sing}} \cup \widehat{f}(W^{F^0}) \cup \widehat{f}(\widehat{Z}_{\text{nd}})$ is a closed subscheme of \overline{X} of dimension $\leq r - 1$. We conclude that $\dim(\text{Sec}(D_1, \mathcal{X}^{F^0} \cup D_2)) \leq r$ for any finite subsets $D_1, D_2 \subset \overline{X}$ of closed points.

Let $T_{D_0, \overline{X}} \subset \mathbb{P}_k^N$ be the union of the tangent spaces to \overline{X} at all points of D_0 . This is a finite union of linear subspaces of dimension r . Thus, for each $x \in D_0$, $\mathcal{Z}_x^{F^0} := \overline{X} \cup T_{D_0, \overline{X}} \cup \text{Sec}(\{x\}, \mathcal{X}^{F^0} \cup (D_0 \setminus \{x\}))$ is a closed subset of \mathbb{P}_k^N of dimension r .

Now by Lemmas 7.2.2 and 7.2.3, the set $\mathcal{U}^{F^0} := \bigcap_{x \in D_0} \text{Gr}(\mathcal{Z}_x^{F^0}, N - r - 1, H_{N,0})$ is dense open in $\text{Gr}(N - r - 1, H_{N,0})$, and for every $L \in \mathcal{U}^{F^0}$, the map $\phi_L : \mathbb{P}_k^N \setminus L \rightarrow \mathbb{P}_k^r$ induces a finite map $\phi_L : X \rightarrow \mathbb{A}_k^r$ which satisfies the properties (2), (3), (5) and (6), and is unramified at D_0 . Since ϕ_L is a finite map of smooth schemes of same dimension, it is also flat by [16, Exercise III-10.9, p.276]. Hence, ϕ_L is étale at D_0 , giving the property (1). The property (4) follows because $\deg(\phi_L) > 1$ by the assumptions on X .

Step 2. *Now suppose that k is any infinite perfect field.* For any k -scheme A , let $\pi_A : A_{\overline{k}} \rightarrow A$ be the base change. Let $D_0 = \{x_1, \dots, x_s, x_{s+1}, \dots, x_p\}$. For $1 \leq i \leq p$, let $S_i = \pi_X^{-1}(x_i) = \{x_i^1, \dots, x_i^{r_i}\}$ and let $S = \bigcup_{i=1}^p S_i$. Since D_0 is a set of smooth points of \overline{X} , we see that S is a set of smooth points of $\overline{X}_{\overline{k}}$. Let $\widehat{Z}_{\overline{k}} = \pi_{X \times \widehat{B}}^{-1}(\widehat{Z})$ and $\overline{F^0} = F_{\overline{k}} \cup \pi_B^{-1}(E^0)$. Notice that no component of $\widehat{Z}_{\overline{k}}$ is contained in $X_{\overline{k}} \times \overline{F^0}$.

Using Step 1, we choose an embedding $\eta : \overline{X} \hookrightarrow \mathbb{P}_k^N$ such that for the inclusion $X_{\overline{k}} \hookrightarrow \mathbb{P}_{\overline{k}}^N$, there is a dense open subset $\mathcal{U} \subset \text{Gr}(N - r - 1, H_{N,0,\overline{k}})$ such that each $L \in \mathcal{U}$ satisfies the assertions of the lemma with $S \subset X_{\overline{k}}$, $\overline{F^0} \subset B_{\overline{k}}$ and $\widehat{Z}_{\overline{k}} \subset X_{\overline{k}} \times_{\overline{k}} \widehat{B}_{\overline{k}}$ chosen as above.

Let $M = \dim \text{Gr}(N - r - 1, H_{N,0,\overline{k}}) = \dim \mathcal{U}$. Since $\text{Gr}(N - r - 1, H_{N,0,\overline{k}})$ contains an affine space $\mathbb{A}_{\overline{k}}^M$ as an open subset, by replacing \mathcal{U} by $\mathcal{U} \cap \mathbb{A}_{\overline{k}}^M$, we may assume $\mathcal{U} \subset \mathbb{A}_{\overline{k}}^M$. Since k is infinite, the set of points in $\mathbb{A}_{\overline{k}}^M$ with coordinates in k is dense in $\mathbb{A}_{\overline{k}}^M$. Hence, there is a dense subset of \mathcal{U} each of whose points L is defined over k , i.e., $L \in \text{Gr}(N - r - 1, H_{N,0})$. Since $\overline{X} \cap L = \emptyset$ by the choice of L , we can apply Lemma 7.2.2 to get a finite map $\phi_L : X \rightarrow \mathbb{A}_k^r$ over k . We show that ϕ_L satisfies the properties (1)~(6) of the lemma.

Since ϕ_L is already flat (by [16, Exercise III-10.9, p.276] being a finite map of smooth schemes), to prove (1), we only need to check that it is unramified at each point of D_0 . So, fix a point $x \in D_0$ and let $y = \phi_L(x)$. Given a point $x' \in \pi_X^{-1}(x)$, let $y' = \phi_{L_{\overline{k}}}(x')$ and $\ell = k(x')$. This yields a commutative diagram of regular local

rings

$$(7.8) \quad \begin{array}{ccc} \mathcal{O}_{\mathbb{A}_k^r, y} & \longrightarrow & \mathcal{O}_{\mathbb{A}_\ell^r, y'} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X, x} & \longrightarrow & \mathcal{O}_{X_\ell, x'}. \end{array}$$

Since k is perfect, the two horizontal maps are étale. We had shown previously that the map $\mathcal{O}_{\mathbb{A}_k^r, y'} \rightarrow \mathcal{O}_{X_k, x'}$ is étale. Equivalently, the map $\mathcal{O}_{\mathbb{A}_\ell^r, y'} \rightarrow \mathcal{O}_{X_\ell, x'}$ is étale. Thus, the composite map $\mathcal{O}_{\mathbb{A}_k^r, y} \rightarrow \mathcal{O}_{X, x} \rightarrow \mathcal{O}_{X_\ell, x'}$ is étale. Now, by Lemma 7.1.1, the map $\mathcal{O}_{\mathbb{A}_k^r, y} \rightarrow \mathcal{O}_{X, x}$ is étale. This proves (1).

Before we go further, we check the following claim. Consider the Cartesian square

$$(7.9) \quad \begin{array}{ccc} X_{\bar{k}} & \xrightarrow{\phi_{L_{\bar{k}}}} & \mathbb{A}_{\bar{k}}^r \\ \pi_X \downarrow & & \downarrow \pi_{\mathbb{A}^r} \\ X & \xrightarrow{\phi_L} & \mathbb{A}_k^r. \end{array}$$

Claim: For a closed point $x \in X$ and $y := \phi_L(x)$, one has $|\pi_{\mathbb{A}^r}^{-1}(y)| \leq |\pi_X^{-1}(x)|$, and the equality holds if and only if $[k(x) : k(y)] = 1$. Furthermore, this equality holds if the map $\phi_{L_{\bar{k}}} : \pi_X^{-1}(x) \rightarrow \pi_{\mathbb{A}^r}^{-1}(y)$ is injective.

(\because) Since k is perfect, we have $|\pi_X^{-1}(x)| = [k(x) : k]$ and $|\pi_{\mathbb{A}^r}^{-1}(y)| = [k(y) : k]$. The inclusions $k \hookrightarrow k(y) \hookrightarrow k(x)$ imply the first assertion. Next, the injectivity of the map $\phi_{L_{\bar{k}}} : \pi_X^{-1}(x) \rightarrow \pi_{\mathbb{A}^r}^{-1}(y)$ implies that $|\pi_{\mathbb{A}^r}^{-1}(y)| \geq |\pi_X^{-1}(x)|$. The second part of the Claim follows now.

We now go back to the proof of the remaining properties of ϕ_L . Since the map $\phi_{L_{\bar{k}}}$ is injective on S , the properties (2) and (3) follow directly from the above Claim and our choice of S .

Since $L^+(S) \cap \widehat{f_{\bar{k}}}(\widehat{Z_{\bar{k}}^{F^0}}) = L^+(S) \cap \widehat{f_{\bar{k}}}((\widehat{Z_{\bar{k}}})_{\text{nd}}) = \emptyset$, the properties (5) and (6) follow, if the following assertion holds:

$$(\star) \quad \pi_X^{-1}(L^+(D_0)) = L^+(S) \text{ and } \pi_X^{-1}(\widehat{f}(\widehat{Z^{F^0}})) = \widehat{f_{\bar{k}}}(\widehat{Z_{\bar{k}}^{F^0}}).$$

To prove (\star) , it is clear from the choice of S that $L^+(S) \subset \pi_X^{-1}(L^+(D_0))$. Suppose now that there is an $x' \in X_{\bar{k}} \setminus S$ such that $\pi_{\mathbb{A}^r} \circ \phi_{L_{\bar{k}}}(x') = \phi_L(x) = y$ for some $x \in D_0$. This implies that $x' \in \phi_{L_{\bar{k}}}^{-1}((\pi_{\mathbb{A}^r})^{-1}(y))$.

On the other hand, it follows from the Claim and our choice of S that $\pi_{\mathbb{A}^r}^{-1}(y) = \phi_{L_{\bar{k}}}(\pi_X^{-1}(x))$. We deduce that $x' \in \phi_{L_{\bar{k}}}^{-1}(\phi_{L_{\bar{k}}}(\pi_X^{-1}(x))) \setminus S$ and hence $x' \in L^+(\pi_X^{-1}(x)) \subset L^+(S)$.

To prove the second equality of (\star) , it is again clear that $\widehat{f_{\bar{k}}}(\widehat{Z_{\bar{k}}^{F^0}}) \subset \pi_X^{-1}(\widehat{f}(\widehat{Z^{F^0}}))$. For the reverse inclusion, let $x' \in X_{\bar{k}}$ be such that $x = \pi_X(x') \in \widehat{f}(\widehat{Z^{F^0}})$. This implies that there is $\alpha = (x, b) \in \widehat{Z^{F^0}}$ with $\widehat{f}(\alpha) = x$ and $b \in F^0$. This implies that $\alpha' = (x', b) \in \pi_{X \times \widehat{B}}^{-1}(\widehat{Z^{F^0}}) = \widehat{Z_{\bar{k}}^{F^0}}$ and $\widehat{f_{\bar{k}}}(\alpha') = x'$. This proves (\star) .

To prove (4), recall that $\phi_L : X \rightarrow \mathbb{A}_k^r$ is finite and flat. By our assumptions on X , it is not an isomorphism. Since ϕ_L is étale at $x \in D_0$ with $k(\phi_L(x)) \simeq k(x)$, the

set $L^+(x)$ must be nonempty, as shown in the proof of Lemma 7.1.3. This proves the property (4), thus completes the proof of the lemma. \square

7.4. The fs-moving for additive cycles. We now apply Lemma 7.3.2 to additive cycles, where we take $B = \mathbb{A}_k^1 \times \square_k^{n-1}$, $\widehat{B} = \widehat{B}_n = \mathbb{P}_k^1 \times \overline{\square}_k^{n-1}$ and $F = F_n = \widehat{B}_n \setminus B_n$. We remark that one can check that exactly the same proof also yields a similar lemma for higher Chow cycles, with $B = \square_k^n$ and $\widehat{B} = \overline{\square}_k^n$.

Proposition 7.4.1. *Let $V = \text{Spec}(R)$ be an r -dimensional regular semi-local k -scheme of geometric type with the set of closed points Σ . For integers $m, n \geq 1$, let $\alpha \in \text{Tz}^n(V, n; m)$ be a cycle.*

Suppose no component of α is an fs-cycle. Suppose further that $r \geq 1$ and that V is not α -linear (see Definition 5.1.3). Then we can find

- (1) *an affine atlas (X, Σ) for V ,*
- (2) *a cycle $\overline{\alpha} \in \text{Tz}^n(X, n; m)$ with $\alpha = \overline{\alpha}_V$.*
- (3) *a finite and flat map $\phi : X \rightarrow \mathbb{A}_k^r$, and*
- (4) *an affine open neighborhood $U \subset X$ of Σ*

such that the following hold:

- (A) *If Z_i is a component of α which is dominant over V , then for every component Z' of $L^+([\overline{Z}_i])$, the map $Z'_U \rightarrow U$ is finite and surjective.*
- (B) *If Z_i is a component of α , not dominant over V , then $L^+([\overline{Z}_i])_U = 0$.*

Proof. Let (X, Σ) be an affine atlas for V such that for a cycle $\overline{\alpha} \in \text{Tz}^n(X, n; m)$, we have $\alpha = \overline{\alpha}_V$. See Lemma 2.4.3. Here, every component $\overline{\alpha}$ is the closure of exactly one component of α in $X \times B_n$. Since V is not α -linear, X is not isomorphic to an affine space over k . Let $\{Z_1, \dots, Z_s\}$ be the set of irreducible components α and let $Z := \text{Supp}(\alpha)$, $\overline{Z} := \text{Supp}(\overline{\alpha}) = \bigcup_{i=1}^s \overline{Z}_i$.

Let \widehat{Z} and \widehat{Z}_i be the closures of \overline{Z} and \overline{Z}_i in $X \times \widehat{B}_n$. Let $f : \overline{Z} \hookrightarrow X \times B_n \rightarrow X$ and $\widehat{f} : \widehat{Z} \hookrightarrow X \times \widehat{B}_n \rightarrow X$ be the composites with the projection maps. Similarly, define $\widehat{g} : \widehat{Z} \rightarrow \widehat{B}_n$. Note the map \widehat{f} is projective.

For given $1 \leq i \leq s$, we fix closed points $x_i \in X$, $b_i \in B$ such that $\alpha_i = (x_i, b_i) \in \overline{Z}_i$ and $\alpha_i \notin \overline{Z}_j$ for $i \neq j$. Since each $\overline{Z}_i \neq \emptyset$, such closed points always exist. Note that \overline{Z} is an r -dimensional cycle in $X \times B_n$, none of whose component is an fs-cycle over V , so in particular, no component of \overline{Z} is contained in $X \times \{b\}$ for any closed point $b \in B_n$ (for otherwise, the component contained in $X \times \{b\}$ is finite surjective over X so that it gives an fs-cycle over V , contradiction to the given assumption). Set $D_0 = \{x_1, \dots, x_s\} \cup \Sigma$, $E^0 = \{b_1, \dots, b_s\}$ and $F^0 = F_n \cup E^0$.

Let $\phi_L : \overline{X} \rightarrow \mathbb{P}_k^r$ and $\phi_L : X \rightarrow \mathbb{A}_k^r$ be the finite maps which satisfy the properties (1)~(6) stated in Lemma 7.3.2 with \overline{Z} , D_0 and F^0 as chosen above. Note that the map $\phi_L : X \rightarrow \mathbb{A}_k^r$ is flat because X is smooth of dimension r using [16, Exercise III-10.9, p.276]. For simplicity, write $\phi = \phi_L$, $\phi_{B_n} = \phi_L \times \text{Id}_{B_n}$, and $\phi_{\widehat{B}_n} = \phi_L \times \text{Id}_{\widehat{B}_n}$.

For an irreducible component \overline{Z}_i of \overline{Z} , note that the irreducible components of $L^+(\overline{Z}_i)$ are exactly the restrictions of the irreducible components of $L^+(\widehat{Z}_i)$ to $X \times B_n$.

Claim 1. *No irreducible component of $L^+(\widehat{Z}_i)$ coincides with \widehat{Z}_i .*

(\cdot): Let $y_i = \phi(x_i)$ and $\beta_i = \phi_{\widehat{B}_n}(\alpha_i) = (y_i, b_i)$. Set

$$Y_i = \widehat{f}(\widehat{Z}_i), Y_i^0 = \widehat{f}(\widehat{Z}_i^{F^0}), W_i = \phi_{B_n}(\overline{Z}_i) \text{ and } \widehat{W}_i = \phi_{\widehat{B}_n}(\widehat{Z}_i).$$

One checks from the definitions of the flat pull-back and proper push-forward of cycles ([27, Theorem 3.1], [26, § 3.3, 3.4]) that \widehat{Z}_i is not a component of $L^+(\widehat{Z}_i)$ if and only if the map $\mathcal{O}_{W_i, \beta_i} \rightarrow \mathcal{O}_{\overline{Z}_i, \beta_i}$ of semi-local rings is an isomorphism. We prove the latter.

It follows from the property (5) of Lemma 7.3.2 that the map $\mathcal{O}_{\overline{Z}_i, \beta_i} \rightarrow \mathcal{O}_{\overline{Z}_i, \alpha_i}$ is an isomorphism. By the property (1) of Lemma 7.3.2, the map ϕ is étale in an affine neighborhood U' of D_0 , and hence $\phi_{\widehat{B}_n}$ is étale in $U' \times \widehat{B}_n$. In particular, it is étale at $\alpha_i \in U' \times \widehat{B}_n$, thus the map $\mathcal{O}_{W_i, \beta_i} \rightarrow \mathcal{O}_{\overline{Z}_i, \alpha_i}$ is unramified. By the property (3) of Lemma 7.3.2 of the map ϕ , we have $k(\beta_i) \xrightarrow{\sim} k(\alpha_i)$. We conclude that $\mathcal{O}_{W_i, \beta_i} \rightarrow \mathcal{O}_{\overline{Z}_i, \beta_i}$ is an injective, finite and unramified map of local rings which induces isomorphism of the residue fields. But, by Lemma 7.1.2, it must be an isomorphism, proving Claim 1.

We now prove (A). Let \overline{Z}_i be an irreducible component of \overline{Z} which is dominant over V . Let Z' be an irreducible component of $L^+(\widehat{Z}_i)$. By Lemma 5.1.5, it is enough to show that $Z' \cap (\Sigma \times F_n) = \emptyset$.

Toward contradiction, suppose there are closed points $x \in \Sigma$ and $b \in F_n$ such that $\lambda = (x, b) \in Z'$. Note that $\lambda = (x, b) \in Z'$ means that there is a point $x' \in \phi^{-1}(\phi(x))$ such that $\lambda' = (x', b) \in \widehat{Z}_i$. If $x' \neq x$, then $x' \in L^+(x)$ and $x' \in Y_i^0$, which cannot happen by the property (5) of Lemma 7.3.2. So, we must have $x' = x$ so that $\lambda = (x, b) \in Z' \cap \widehat{Z}_i$. We show this cannot happen either.

Let $\xi = \phi_{\widehat{B}_n}(\lambda) = (\phi(x), b) = (y, b)$. Let $S := (\phi_{\widehat{B}_n})^{-1}(\xi) = \phi^{-1}(y) \times \{b\}$. Since $\phi_{\widehat{B}_n}$ is étale in $U' \times \widehat{B}_n$, it is étale at λ . By the property (3) of Lemma 7.3.2 of the map ϕ , we have $k(\xi) \xrightarrow{\sim} k(\lambda)$.

Claim 2. $\widehat{Z}_i \cap S = \{\lambda\}$.

(\cdot): If not, then there is a point $\lambda' = (x', b) \in \widehat{Z}_i$ with $\lambda' \in S \setminus \{\lambda\}$. Then we must have $\lambda' \in \widehat{Z}_i^{F_n}$ and $x' \in L^+(x)$. In particular, $x' \in Y_i^0 \cap L^+(x)$. But, again by the property (5) of Lemma 7.3.2 of the map ϕ , this is a contradiction, so we have Claim 2.

Now, that $\lambda \in Z' \cap \widehat{Z}_i$ and Claim 2 together contradict Lemma 7.1.3. Hence, $Z' \cap (\Sigma \times F_n) = \emptyset$ as desired, proving (A).

We prove (B) now. Suppose next that \overline{Z}_i is an irreducible component of \overline{Z} which is not dominant over V . Suppose that Z' is some component of $L^+(\widehat{Z}_i)$ such that $Z' \cap (\Sigma \times \widehat{B}_n) \neq \emptyset$. Then we can find $x \in \Sigma$ and $b \in \widehat{B}_n$ such that $\lambda = (x, b) \in Z'$. This means that there is a point $x' \in \phi^{-1}(\phi(x))$ such that $\lambda' = (x', b) \in \widehat{Z}_i$. If $x' \in L^+(x)$, then we must have $x' \in L^+(x) \cap Y_i$, which is empty as before. If $x' = x$, then we must have $\lambda = (x, b) \in Z' \cap \widehat{Z}_i$. But the same proof as above shows that this can not occur. We conclude that $\widehat{f}(L^+(\widehat{Z}_i))$ is a closed subset of X disjoint from Σ . Hence we can choose an affine open neighborhood U of Σ in X such that $L^+((\widehat{Z}_i)_U) = \emptyset$. This proves (B), and proof of the proposition is now complete. \square

Theorem 7.4.2 (The fs-moving lemma). *Let $V = \operatorname{Spec}(R)$ be a regular semi-local k -scheme of geometric type with the set of closed points Σ . Suppose that $r = \dim(V) \geq 1$ and let $m, n \geq 1$. Then the map $\operatorname{fs}_V : \operatorname{TCH}_{\operatorname{fs}}^n(V, n; m) \rightarrow \operatorname{TCH}^n(V, n; m)$ is an isomorphism.*

Proof. From the definition of $\operatorname{TCH}_{\operatorname{fs}}^n(V, n; m)$, the map fs_V is injective. We prove surjectivity. Let $\gamma \in \operatorname{Tz}^n(V, n; m)$ be a cycle with $\partial(\gamma) = 0$. Write $\gamma = \alpha + \beta$, where no component of α is an fs-cycle and every component of β is an fs-cycle.

First suppose that V is α -linear, so there is an atlas (\mathbb{A}_k^r, Σ) for α . In this case, by Theorem 6.2.1, we can write $\alpha = \alpha_1 + \partial(\alpha_2)$, where $\alpha_1 \in \operatorname{Tz}_{\operatorname{sfs}}^n(V, n; m) \subset \operatorname{Tz}_{\operatorname{fs}}^n(V, n; m)$ and $\alpha_2 \in \operatorname{Tz}^n(V, n+1; m)$. In particular, $\gamma = \alpha_1 + \beta + \partial(\alpha_2)$. So, for $\gamma' := \alpha_1 + \beta \in \operatorname{Tz}_{\operatorname{fs}}^n(V, n; m)$, one immediately has $\partial(\gamma') = 0$ and $\gamma - \gamma' = \partial(\alpha_2)$, proving the theorem in this case.

Now suppose V is not α -linear. By Lemma 2.4.3, there is an affine atlas (X, Σ) for V , and cycles $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \operatorname{Tz}^n(X, n; m)$ such that $\bar{\alpha}_V = \alpha$, $\bar{\beta}_V = \beta$, $\bar{\gamma}_V = \gamma$, $\bar{\gamma} = \bar{\alpha} + \bar{\beta}$ and $\partial(\bar{\gamma}) = 0$, and X is not isomorphic to \mathbb{A}_k^r . Moreover, by applying Proposition 7.4.1 to α , we may assume $\phi : X \rightarrow \mathbb{A}_k^r$ is a finite flat map as in *loc.cit.*, and α satisfies all the properties there. Let $\Sigma' = \phi(\Sigma)$, which consists of finitely many closed points of \mathbb{A}_k^r . Let $V' = \operatorname{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \Sigma'})$ and $W := X \times_{\mathbb{A}_k^r} V'$. Here there are inclusions $\Sigma \subset V \subset W \subset X$, and a finite flat morphism $\phi : W \rightarrow V'$. Furthermore, V' is $\phi_*(\gamma)$ -linear by definition.

Write $\bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2$, where each component of $\bar{\alpha}_1$ is dominant over X and no component of $\bar{\alpha}_2$ is dominant over X . So, we can write

$$\bar{\gamma} = \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta} = (\bar{\alpha}_1 - \phi^* \phi_*(\bar{\alpha}_1)) + (\bar{\alpha}_2 - \phi^* \phi_*(\bar{\alpha}_2)) + (\bar{\beta} - \phi^* \phi_*(\bar{\beta})) + \phi^* \phi_*(\bar{\gamma}).$$

Let $\bar{\alpha}'_i := \bar{\alpha}_i - \phi^* \phi_*(\bar{\alpha}_i)$ for $i = 1, 2$, and $\bar{\beta}' := \bar{\beta} - \phi^* \phi_*(\bar{\beta})$. Note that ϕ_* and ϕ^* preserve the fs-cycles so that $\bar{\beta}'$ is an fs-cycle. On the other hand, by Proposition 7.4.1(B), we have $(\bar{\alpha}'_2)_V = 0$, while by Proposition 7.4.1(A) we have $(\bar{\alpha}'_1)_V \in \operatorname{Tz}_{\operatorname{fs}}^n(V, n; m)$. Finally, since $\bar{\gamma} \in \operatorname{Tz}^n(X, n; m)$ with $\partial(\bar{\gamma}) = 0$, we have $\phi_*(\bar{\gamma}) \in \operatorname{Tz}^n(\mathbb{A}_k^r, n; m)$ with $\partial(\phi_*(\bar{\gamma})) = 0$. Furthermore, V' is $\phi_*(\gamma)$ -linear, so by Theorem 6.2.1, there are cycles $\eta_1 \in \operatorname{Tz}_{\operatorname{fs}}^n(V', n; m)$, and $\eta_2 \in \operatorname{Tz}^n(V', n+1; m)$ such that $j^*(\phi_*(\bar{\gamma})) = \eta_1 + \partial\eta_2$. This is equivalent to $\phi_*(\bar{\gamma}_W) = \eta_1 + \partial\eta_2$. Hence, $\phi^* \phi_*(\bar{\gamma}_W) = \phi^*(\eta_1) + \phi^*(\partial\eta_2) = \phi^*(\eta_1) + \partial(\phi^*\eta_2)$. Note $\phi^* \phi_*(\bar{\gamma}_W)_V = \phi^* \phi_*(\bar{\gamma})_V$.

Hence, combining these, we have $\gamma = (\bar{\gamma})_V = (\bar{\alpha}'_1)_V + \bar{\beta}'_V + (\phi^*(\eta_1))_V + \partial((\phi^*\eta_2)_V)$, where $\gamma_1 := (\bar{\alpha}'_1)_V + \bar{\beta}'_V + (\phi^*(\eta_1))_V \in \operatorname{Tz}_{\operatorname{fs}}^n(V, n; m)$. Since $\partial\gamma = 0$, we also deduce $\partial\gamma_1 = 0$. This complete the proof. \square

8. THE sfs-MOVING LEMMA I: ADMISSIBLE LINEAR SUBSPACES

Let k be an infinite perfect field. From §8.2 to the end of §9, we suppose k is algebraically closed. The goal of the next a few sections is to show that the map $\operatorname{sfs}_V : \operatorname{TCH}_{\operatorname{sfs}}^n(V, n; m) \rightarrow \operatorname{TCH}_{\operatorname{fs}}^n(V, n; m)$ is an isomorphism for a regular semi-local k -scheme V of geometric type. As for the fs-moving lemma in §7, we use techniques of linear projections inside projective and affine spaces to reduce to the case when V is linear with respect to a given admissible cycle. In this section, we define certain admissible linear subspaces and loci in a Grassmannian space and prove some results on them. These results contribute to the final proof of the sfs-moving

lemma. For the definition and some elementary properties of sfs-cycles, see §5. We use the notations and terminologies of §7.2 on linear projections.

8.1. Rectifiable cycles. Let X be an irreducible quasi-projective scheme over k of dimension $r \geq 1$. Let $X_{\text{fs}} \subset X$ be a fixed dense open subset which is smooth over k (possibly smaller than X_{sm}) and let X_{nfs} be its complement. Notice that $X_{\text{sing}} \subset X_{\text{nfs}}$. Let B be a geometrically integral smooth affine scheme over k of positive dimension, and let \widehat{B} be a geometrically integral smooth projective compactification of B . Let Z be a cycle on $X \times \widehat{B}$ and let $\{Z_1, \dots, Z_s\}$ be its irreducible components. Let $f : Z \rightarrow X$ and $g : Z \rightarrow \widehat{B}$ be the restrictions of the projection maps.

Definition 8.1.1. We say that Z is a *rectifiable cycle*, if the following hold for each irreducible component Z_i of Z :

- (1) the map $Z_i \rightarrow X$ is finite and surjective over X_{fs} , and
- (2) the map $Z_i \rightarrow \widehat{B}$ is not constant.

When no confusion arises, we will say that Z is an fs-cycle (over X_{fs}), if each Z_i satisfies (1), but not necessarily (2). Certainly this notion depends on the choice of X_{fs} , but it is invariant under shrinking X_{fs} once it is an fs-cycle.

8.2. Admissible linear subspaces. We assume for the rest of §8 that the ground field k is algebraically closed.

Suppose X is projective and fix a closed embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ such that $N \gg r$ and of degree $d + 1 \gg 0$. We fix X_{fs} and fix a closed point $x \in X_{\text{fs}}$. Let Z be a rectifiable cycle on $X \times \widehat{B}$ with irreducible components $\{Z_1, \dots, Z_s\}$. Let $f : Z_i \rightarrow X$ and $g : Z_i \rightarrow \widehat{B}$ be the projections. Let $S \subset X \setminus \{x\}$ be a given finite set of closed points.

Definition 8.2.1. We say that a linear subspace $L \in \text{Gr}(N - r, \mathbb{P}_k^N)$ is (Z, x) -*admissible*, if the following hold:

- (1) $L \cap (X_{\text{nfs}} \cup S) = \emptyset$.
- (2) L intersects X_{fs} transversely.
- (3) $L \cap X = \{x = x_0, x_1, \dots, x_d\}$ with $x_i \neq x_j$ for $i \neq j$.
- (4) Each Z_i is regular at all points lying over $\{x_1, \dots, x_d\}$.
- (5) $g(Z_{x_i}) \cap g(Z_{x_j}) = \emptyset$ for $0 \leq i \neq j \leq d$.

Note that $L \in \text{Gr}(N - r, \mathbb{P}_k^N)$ is (Z, x) -admissible if and only if it is (Z_i, x) -admissible for every irreducible component Z_i of Z . The following is an “intermediate” version of Definition 8.2.1:

Definition 8.2.2. For $1 \leq n \leq d - 1$ and $0 \leq m \leq n$, we say that a member $\underline{L} = (L, x_1, \dots, x_d) \in \text{Gr}(N - r, \mathbb{P}_k^N) \times X^d$ is (Z, x, m, n) -*admissible*, if the following hold:

- (1) $L \cap (X_{\text{nfs}} \cup S) = \emptyset$.
- (2) L intersects X_{fs} transversely.
- (3) $L \cap X = \{x = x_0, x_1, \dots, x_d\}$ with $x_i \neq x_j$ for $i \neq j$.
- (4) Each Z_i is regular at all points lying over $\{x_1, \dots, x_d\}$.
- (5) $g(Z_{x_i}) \cap g(Z_{x_j}) = \emptyset$ for $0 \leq i \neq j \leq n$.

$$(6) \quad g(Z_{x_i}) \cap g(Z_{x_{n+1}}) = \emptyset \text{ for } 0 \leq i \leq m.$$

Note that \underline{L} is (Z, x, m, n) -admissible if and only if it is (Z_i, x, m, n) -admissible for every irreducible component Z_i of Z . Note also that $L \in \text{Gr}(N - r, \mathbb{P}^N)$ is (Z, x) -admissible if and only if \underline{L} is $(Z, x, d - 1, d - 1)$ -admissible. We now define the *admissible locus* of the simplest kind:

Definition 8.2.3. Let $\widetilde{\mathcal{U}}_{m,n}^{x,S} \subset \text{Gr}_x(N - r, \mathbb{P}_k^N) \times X^d$ be the subset parameterizing all (Z, x, m, n) -admissible points and let $\mathcal{U}_{m,n}^{x,S} \subset \text{Gr}_x(N - r, \mathbb{P}_k^N)$ be the image of $\widetilde{\mathcal{U}}_{m,n}^{x,S}$ under the projection $\text{Gr}_x(N - r, \mathbb{P}_k^N) \times X^d \rightarrow \text{Gr}_x(N - r, \mathbb{P}_k^N)$. Let $\mathcal{U}_{\text{adm}}^{x,S} \subset \text{Gr}_x(N - r, \mathbb{P}_k^N)$ be the subset parameterizing all (Z, x) -admissible points.

Before we proceed to the relative notion of admissibility, we prove the following:

Lemma 8.2.4. *Suppose $r = 1$, $N \gg r$ and let $x \neq y$ be two closed points on X_{sm} . Let $\text{Gr}_{x+2y}(N - 1, \mathbb{P}_k^N) \subset \text{Gr}(N - 1, \mathbb{P}_k^N)$ be the set of hyperplanes containing $\{x, y\}$ that do not intersect X transversely at y . Then $\text{Gr}_{x+2y}(N - 1, \mathbb{P}_k^N) \simeq \mathbb{P}_k^{N-3}$.*

Proof. Since $N \gg r = 1$, we can find a linear form $s_1 \in W = H^0(\mathbb{P}_k^N, \mathcal{O}(1))$ which does not vanish at $\{x, y\}$. This yields a k -linear map $\alpha : W \rightarrow \mathcal{O}_{X, \{x, y\}} / \mathfrak{m}_x \mathfrak{m}_y^2$ given by $\alpha(s) = s/s_1$. Since k is algebraically closed and so \mathfrak{m}_y is generated by linear forms vanishing at y , we see that the composite map $W \rightarrow \mathcal{O}_{X, \{x, y\}} / \mathfrak{m}_x \mathfrak{m}_y^2 \rightarrow \mathcal{O}_{X, y} / \mathfrak{m}_y^2$ is surjective and $\alpha^{-1}(\mathfrak{m}_y^2)$ is precisely the set of linear forms in W not transverse to X at y .

Since x, y are two distinct smooth closed points of X , the set $\text{Gr}_y(x, N - 1, \mathbb{P}_k^N)$ is nonempty and hence $\mathfrak{m}_y / \mathfrak{m}_x \mathfrak{m}_y \xrightarrow{\sim} \mathcal{O}_{\{x\}}$ and there is a commutative diagram of short exact sequences:

$$(8.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & \alpha^{-1}(\mathfrak{m}_x \mathfrak{m}_y) & \rightarrow & \alpha^{-1}(\mathfrak{m}_y) & \rightarrow & \mathcal{O}_{\{x\}} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \mathfrak{m}_x \mathfrak{m}_y / \mathfrak{m}_x \mathfrak{m}_y^2 & \rightarrow & \mathfrak{m}_y / \mathfrak{m}_y^2 & \rightarrow & \mathcal{O}_{\{x\}} \rightarrow 0. \end{array}$$

In particular, the first vertical map is surjective. Since $\text{Gr}_x(y, N - 1, \mathbb{P}_k^N) \neq \emptyset$, we conclude that α is surjective. We now have a commutative diagram of short exact sequences

$$(8.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & W(-x - 2y) & \rightarrow & W & \rightarrow & \mathcal{O}_{\{x+2y\}} \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \rightarrow & \alpha^{-1}(\mathfrak{m}_y^2) & \rightarrow & W & \rightarrow & \mathcal{O}_{\{2y\}} \rightarrow 0. \end{array}$$

Since the last vertical arrow is surjective with one-dimensional kernel, it follows that the first vertical arrow is injective with one-dimensional cokernel. Since $|\alpha^{-1}(\mathfrak{m}_y^2)| \simeq \mathbb{P}_k^{N-2}$, we conclude that $\text{Gr}_{x+2y}(N - 1, \mathbb{P}_k^N) \simeq |W(-x - 2y)| \simeq \mathbb{P}_k^{N-3}$. \square

Lemma 8.2.5. *Suppose $r = 1$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in X . Then the set $\mathcal{U}_S^{x, 1-4} \subset \text{Gr}_x(N - 1, \mathbb{P}_k^N)$ consisting of hyperplanes H such*

that a member $\underline{H} \in \mathrm{Gr}_x(N-1, \mathbb{P}_k^N) \times X^d$ satisfies the conditions (1) \sim (4) of Definition 8.2.2, is a dense open subset of $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$.

Proof. Since $f : Z \rightarrow X$ is finite over X_{fs} and since $\dim(Z_{\mathrm{sing}}) = 0$, we see that $f(Z_{\mathrm{sing}})$ is a finite closed subset of X . Since $\dim(X) = \dim(X_{\mathrm{fs}}) = r = 1$, we have $|X_{\mathrm{nfs}}| = |X_{\mathrm{fs}}^c| < \infty$. Hence, $T = (f(Z_{\mathrm{sing}}) \cup X_{\mathrm{nfs}} \cup S) \setminus \{x\}$ is a finite closed subset of X . Hence, the hyperplanes not intersecting T form an open subset of $\mathrm{Gr}(N-1, \mathbb{P}_k^N)$. Note also that the set of hyperplanes in $\mathrm{Gr}(N-1, \mathbb{P}_k^N)$ which do not intersect T and possibly intersect X transversely along X_{sm} form an open subset of $\mathrm{Gr}(N-1, \mathbb{P}_k^N)$. We conclude that the hyperplanes H satisfying the conditions (1) \sim (4) of Definition 8.2.2 form an open subset of $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$. We only need to show that this set is nonempty.

For any $y \in X_{\mathrm{sm}}$, let \mathfrak{m}_y denote the maximal ideal of $\mathcal{O}_{X,y}$. Let V be the set of linear forms of $W = H^0(\mathbb{P}_k^N, \mathcal{O}(1))$ that vanish at x . Note $\dim |V| = N-1$. Since $x \in X_{\mathrm{sm}}$, we see that \mathfrak{m}_x is generated by the elements of V . In particular, if we choose s_1 not vanishing at x , then the map $\alpha : V \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$ given by $\alpha(s) = s/s_1$ is surjective and $D_x = \mathbb{P}(\ker(\alpha))$ is exactly the set of hyperplanes passing through x , but not transverse to X at x . It is clear that $\dim(D_x) = N-2 < \dim(\mathrm{Gr}_x(N-1, \mathbb{P}_k^N))$.

Let $B' \subset X \times |V|$ be the set of pairs (y, H) such that H passes through y , but not transverse to X at y . Let's first study the fiber of $\pi_1 : B' \rightarrow X$ over each $y \in X_{\mathrm{sm}} \setminus \{x\}$. Choose $s_1 \in V$ such that $s_1(x) = 0$ but $s_1(y) \neq 0$. Consider the map $\beta : V \rightarrow \mathcal{O}_{X,y}/\mathfrak{m}_y^2$ given by $\beta(s) = s/s_1$. Since $\dim |V| = N-1$, while $\mathrm{Gr}_{\{x,y\}}(N-1, \mathbb{P}^N) \simeq \mathbb{P}^{N-2}$, and $\mathrm{Gr}_{\{x+2y\}}(N-1, \mathbb{P}^N) \simeq \mathbb{P}^{N-3}$ by Lemma 8.2.4, we see that β is surjective and $\mathbb{P}(\ker(\beta)) = \pi_1^{-1}(y)$ has dimension at most $N-3$, because $\dim_k(\mathcal{O}_{X,y}/\mathfrak{m}_y^2) = 2$.

Thus, $\dim(B') \leq \dim X + \dim(\pi_1^{-1}(y)) \leq 1 + N - 3 = N - 2$. Taking the image of the projection $\pi_2 : B' \rightarrow |V|$, we see that the set of hyperplanes in $\mathrm{Gr}_x(N-1, \mathbb{P}^N)$ that are not transverse to X at its smooth intersection points form a proper closed subset. Since $N \gg 0$ and T is finite, the set of hyperplanes that pass through x , but not through T is a dense open subset of $\mathrm{Gr}_x(N-1, \mathbb{P}^N)$. Hence, we have shown that $\mathcal{U}_S^{x,1-4} \subset \mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$ is a dense open subset. \square

8.3. Admissibility relative to a linear subspace. Now suppose $r \geq 2$, and let $L_0 \in \mathrm{Gr}(N-r+1, \mathbb{P}_k^N)$ be a fixed $(N-r+1)$ -dimensional linear subspace. A hyperplane in \mathbb{P}_k^N intersects L_0 in a linear subspace of dimension at least $N-r$. The subset $\mathrm{Gr}^{\mathrm{tr}}(L_0, N-1, \mathbb{P}_k^N) \subset \mathrm{Gr}(N-1, \mathbb{P}_k^N)$ of hyperplanes intersecting L_0 transversely, is open and its complement is isomorphic to \mathbb{P}_k^{r-2} . Since $\dim \mathrm{Gr}(N-1, \mathbb{P}^N) = N > r-2$, $\mathrm{Gr}^{\mathrm{tr}}(L_0, N-1, \mathbb{P}_k^N)$ is a dense open subset of $\mathrm{Gr}(N-1, \mathbb{P}_k^N)$.

Definition 8.3.1. Under the above notations, define the regular map of schemes

$$(8.3) \quad \theta_{L_0} : \mathrm{Gr}^{\mathrm{tr}}(L_0, N-1, \mathbb{P}_k^N) \rightarrow \mathrm{Gr}(N-r, L_0)$$

given by $\theta_{L_0}(H) = H \cap L_0$.

One checks that θ_{L_0} is a surjective smooth morphism of relative dimension $r-1$. Here is a relative version of Definition 8.2.2:

Definition 8.3.2. Let $S \subset X \setminus \{x\}$ be a finite set of closed points. For $1 \leq n \leq d-1$ and $0 \leq m \leq n$, we say that $\underline{H} = (H, x_1, \dots, x_d) \in \text{Gr}^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \times X^d$ is (Z, x, m, n) -admissible relative to L_0 if $(\theta_{L_0}(H), x_1, \dots, x_d) \in \text{Gr}(N-r, L_0) \times X^d \hookrightarrow \text{Gr}(N-r, \mathbb{P}_k^N) \times X^d$ is (Z, x, m, n) -admissible.

We say that $H \in \text{Gr}^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ is (Z, x) -admissible relative to L_0 if there exists $\underline{H} = (H, x_1, \dots, x_d) \in \text{Gr}^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \times X^d$ which is $(Z, x, d-1, d-1)$ -admissible relative to L_0 .

We see that \underline{H} is (Z, x, m, n) -admissible relative to L_0 if and only if it is (Z_i, x, m, n) -admissible relative to L_0 for every irreducible component Z_i of Z . Here is a relative version of the admissible loci of Definition 8.2.3:

Definition 8.3.3. Recall $\text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) := \text{Gr}^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \cap \text{Gr}_x(N-1, \mathbb{P}_k^N)$. Let $\widetilde{\mathcal{U}_{m,n}^{x,S,L_0}} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \times X^d$ be the set of all points that are (Z, x, m, n) -admissible relative to L_0 and let $\mathcal{U}_{m,n}^{x,S,L_0} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ be the image of $\widetilde{\mathcal{U}_{m,n}^{x,S,L_0}}$ under the projection $\text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \times X^d \rightarrow \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$. Let $\mathcal{U}_{\text{adm}}^{x,S,L_0} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ be the set of all points that are (Z, x) -admissible relative to L_0 .

Before we move on to the proof of openness of admissible loci $\mathcal{U}_{m,n}^{x,S}$ and $\mathcal{U}_{m,n}^{x,S,L_0}$, we prove the following higher dimensional analogue of Lemma 8.2.5.

Lemma 8.3.4. Suppose $r \geq 2$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with a Veronese embedding, one has the following.

Given a hyperplane $H_0 \subset \mathbb{P}^N$ disjoint from $S \cup \{x\}$, and a general $L_0 \in \text{Gr}^{\text{tr}}(H_0, N-r+1, \mathbb{P}_k^N)$, the set $\mathcal{U}_S^{x,L_0,1-4} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ consisting of hyperplanes H such that a member $\underline{H} \in \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \times X^d$ satisfies the conditions (1) \sim (4) of Definition 8.3.2, is a dense open subset of $\text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$.

Proof. By the Bertini theorems of Altman and Kleiman [1], a general intersection of $(r-1)$ hypersurfaces containing $S \cup \{x\}$, all of a fixed degree in \mathbb{P}^N (depending only on X and S), is an irreducible curve C . This curve C contains $S \cup \{x\}$; C is not contained in $f(Z_{\text{sing}}) \cup X_{\text{nfs}}$; C is smooth away from X_{sing} , and the map $f^{-1}(C) \rightarrow \widehat{B}$ is not constant.

Hence, after replacing the embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with the Veronese embedding of \mathbb{P}_k^N given by the above degree of hypersurfaces, we can find an $(r-1)$ -tuple of hyperplanes (H_1, \dots, H_{r-1}) , each of which is from $\text{Gr}_{S \cup \{x\}}(N-1, \mathbb{P}_k^N)$, such that the linear subspace $L_0 = H_1 \cap \dots \cap H_{r-1}$ has the following property: L_0 is transverse to H_0 , $C = L_0 \cap X$ is an irreducible curve such that $C \not\subset f(Z_{\text{sing}}) \cup X_{\text{nfs}}$, C is smooth away from X_{sing} and the map $f^{-1}(C) \rightarrow \widehat{B}$ is not constant. Moreover, any general $(r-1)$ -tuple of hyperplanes (H_1, \dots, H_{r-1}) , each from $\text{Gr}_{S \cup \{x\}}(N-1, \mathbb{P}_k^N)$, has this property. Set $S' = (C \setminus \{x\}) \cap (f(Z_{\text{sing}}) \cup X_{\text{nfs}} \cup S)$. The choice of C implies that S' is a finite closed subset of $C \setminus \{x\}$.

It follows from the definition of the degree of the embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ that a general hyperplane in L_0 will intersect C at $d+1$ points. It follows from Lemma 8.2.5 that the set $\mathcal{U}_{S'}^{x,1-4} \subset \text{Gr}_x(N-r, L_0)$ consisting of hyperplanes $L \subset L_0$ such that $L \cap S' = \emptyset$ and a member $\underline{L} \in \text{Gr}_x(N-r, L_0) \times C^d$ satisfies the conditions

(1) \sim (4) of Definition 8.2.2, is a dense open subset of $\mathrm{Gr}_x(N - r, L_0)$. Since θ_{L_0} is a smooth and surjective morphism such that $\theta_{L_0}^{-1}(\mathrm{Gr}_x(N - r, L_0)) = \mathrm{Gr}_x^{\mathrm{tr}}(L_0, N - 1, \mathbb{P}_k^N)$, we see that $\theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,1-4})$ is a dense open subset of $\mathrm{Gr}_x^{\mathrm{tr}}(L_0, N - 1, \mathbb{P}_k^N)$. Thus, it remains to show that $\theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,1-4}) = \mathcal{U}_S^{x,L_0,1-4}$. From the definition of $\mathcal{U}_S^{x,L_0,1-4}$ and the choice of $\mathcal{U}_{S'}^{x,1-4}$, we have $\mathcal{U}_S^{x,L_0,1-4} \subset \theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,1-4})$. We need to show the opposite inclusion.

Suppose $H \in (\theta_{L_0})^{-1}(\mathcal{U}_{S'}^{x,1-4})$. This means that $\theta_{L_0}(H) \cap S' = \emptyset$ and $\theta_{L_0}(H) = H \cap L_0$ satisfies (1) \sim (4) of Definition 8.2.2 with Z replaced by $f^{-1}(C)$. Since $\theta_{L_0}(H) \cap (X_{\mathrm{nfs}} \cup S) = H \cap (L_0 \cap X) \cap (X_{\mathrm{nfs}} \cup S) = \theta_{L_0}(H) \cap C \cap (X_{\mathrm{nfs}} \cup S) \subset \theta_{L_0}(H) \cap S'$, and since $x \in X_{\mathrm{fs}}$, we see that $\theta_{L_0}(H) \cap (X_{\mathrm{nfs}} \cup S) = \emptyset$.

Since H intersects L_0 transversely, which in turn intersects X transversely along X_{sm} , we see that $\theta_{L_0}(H)$ intersects X transversely along X_{fs} . Also, $\theta_{L_0}(H) \cap X = \theta_{L_0}(H) \cap C = \{x = x_0, x_1, \dots, x_d\}$ with $x_i \neq x_j$ for $i \neq j$. Finally, since $(C \cap f(Z_{\mathrm{sing}})) \setminus \{x\} \subset S'$ and since $\theta_{L_0}(H) \cap S' = \emptyset$, we see that Z is smooth along all points lying over x_i for $1 \leq i \leq d$. This shows that $\theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,1-4}) \subset \mathcal{U}_S^{x,L_0,1-4}$ and completes the proof of the lemma. \square

9. THE sfs-MOVING LEMMA II: OPENNESS OF ADMISSIBLE LOCI

We continue to assume in this section that k is algebraically closed. We study the sets $\mathcal{U}_{m,n}^{x,S}$ and $\mathcal{U}_{m,n}^{x,S,L_0}$. The main interest in §9.1 is to show that they form dense open subsets of various parameter spaces. Later in §9.2, we consider the more general situation where $\{x\}$ is replaced by a finite set of smooth closed points of X .

9.1. Openness of admissible loci I. We use the notations and assumptions of §8.2. Namely, let $\eta : X \hookrightarrow \mathbb{P}_k^N$ be a closed embedding such that $N \gg r = \dim X$ and of degree $d + 1 \gg 0$, and let $x \in X$ be a fixed smooth closed point. Let $S \subset X \setminus \{x\}$ be a finite set of closed points, and let Z be a rectifiable cycle on $X \times \widehat{B}$ as in Definition 8.1.1 with irreducible components $\{Z_1, \dots, Z_s\}$. In the following, Lemmas 9.1.1, 9.1.2, 9.1.3 and Proposition 9.1.4 are about $\mathcal{U}_{m,n}^{x,S}$ when $r = 1$, and Proposition 9.1.5 is about $\mathcal{U}_{m,n}^{x,S,L_0}$ when $r \geq 2$.

Lemma 9.1.1. *Suppose $r = 1$. For any $1 \leq n \leq d - 1$ and $0 \leq m \leq n$, the set $\mathcal{U}_{m,n}^{x,S} \subset \mathrm{Gr}_x(N - 1, \mathbb{P}_k^N)$ of Definition 8.2.3 is open.*

Proof. We have shown in Lemma 8.2.5 that the set $\mathcal{U}_S^{x,1-4} \subset \mathrm{Gr}_x(N - 1, \mathbb{P}_k^N)$ is open. Let $X \xleftarrow{f} Z \xrightarrow{g} \widehat{B}$ be the projection maps. Set $A = g(Z_x)$ and $A_Z = g^{-1}(A)$.

Claim 1: *The morphisms f and g are finite. The sets A and A_Z are finite subsets of closed points of \widehat{B} and Z , respectively.*

(\because) Since Z is rectifiable, in particular it is an fs-cycle, so f is quasi-finite. But, \widehat{B} is projective, so f is a finite morphism. On the other hand, since $\dim(Z) = \dim(X) = r = 1$ and g is not constant on each component Z_i (being rectifiable), the morphism g is quasi-finite. Since X is projective, the morphism g is thus a quasi-finite projective morphism, thus a finite morphism. This proves the first assertion. Because $f : Z \rightarrow X$ is finite, we have $|Z_x| < \infty$. Hence $A = g(Z_x)$ is finite. But, g is finite, we have $A_Z = g^{-1}(A)$ is finite. This proves Claim 1.

Let $V_d \subset X^d$ be the open subset defined by

$$(9.1) \quad V_d = \{(y_1, \dots, y_d) \mid y_i \neq y_j \text{ for } 1 \leq i \neq j \leq d \text{ and } y_i \neq x \text{ for } 1 \leq i \leq d\}.$$

Let $D_1, D_2 \subset V_d$ be given by

$$(9.2) \quad \begin{cases} D_1 = \{(y_1, \dots, y_d) \in V_d \mid g(Z_{y_i}) \cap g(Z_{y_j}) \neq \emptyset \text{ for some } 1 \leq i \neq j \leq n\}, \\ D_2 = \{(y_1, \dots, y_d) \in V_d \mid g(Z_{y_i}) \cap g(Z_x) \neq \emptyset \text{ for some } 1 \leq i \leq n\}. \end{cases}$$

For $0 \leq i \leq m$, let $G_i \subset V_d$ be given by

$$(9.3) \quad \begin{cases} G_0 = \{(y_1, \dots, y_d) \in V_d \mid g(Z_x) \cap g(Z_{y_{n+1}}) \neq \emptyset\}, \\ G_i = \{(y_1, \dots, y_d) \in V_d \mid g(Z_{y_i}) \cap g(Z_{y_{n+1}}) \neq \emptyset \text{ for } 1 \leq i \leq m\}. \end{cases}$$

Claim 2: D_1, D_2 and G_i , for $0 \leq i \leq m$, are closed subsets of V_d .

(\cdot) Let $E_1, E_2 \subset \widehat{B}^d$ be the closed subsets given by

$$(9.4) \quad \begin{cases} E_1 = \{(b_1, \dots, b_d) \in \widehat{B}^d \mid b_i = b_j \text{ for some } 1 \leq i \neq j \leq n\}, \\ E_2 = \{(b_1, \dots, b_d) \in \widehat{B}^d \mid b_i \in A \text{ for some } 1 \leq i \leq n\}. \end{cases}$$

For $0 \leq i \leq m$, let $F_i \subset \widehat{B}^d$ be the closed subsets given by

$$(9.5) \quad \begin{cases} F_0 = \{(b_1, \dots, b_d) \in \widehat{B}^d \mid b_{n+1} \in A\}, \\ F_i = \{(b_1, \dots, b_d) \in \widehat{B}^d \mid b_i = b_{n+1}\} \text{ for } 1 \leq i \leq m. \end{cases}$$

One checks that $D_1 = V_d \cap f^d((g^d)^{-1}(E_1))$, $D_2 = V_d \cap f^d((g^d)^{-1}(E_2))$, and $G_i = V_d \cap f^d((g^d)^{-1}(F_i))$ for $0 \leq i \leq m$. Since $f : Z \rightarrow X$ is finite by Claim 1, one sees that D_1, D_2, G_i are all closed in V_d . This proves Claim 2.

Let $\pi : X^d \rightarrow \text{Sym}^d(X)$ be the quotient map under the action of the symmetric group \mathfrak{S}_d on X^d . Note that \mathfrak{S}_d acts freely on the open subset $V_d \subset X^d$. In particular, the map $V_d \rightarrow \pi(V_d)$ is a finite étale map of degree $d!$. Let $\mathcal{U}_S^{x,1-4} \rightarrow \text{Sym}^d(X)$ be the map $H \mapsto \sum_{i=1}^d [y_i]$, where $H \cap X = \{x = x_0, y_1, \dots, y_d\}$. The property (3) in Definition 8.2.2 implies that the image of $\mathcal{U}_S^{x,1-4}$ under this map lies in $\pi(V_d)$. Define $\mathcal{V}_S^{x,1-4}$ by the Cartesian diagram

$$(9.6) \quad \begin{array}{ccc} \mathcal{V}_S^{x,1-4} & \xrightarrow{e} & V_d \\ \psi \downarrow & & \downarrow \pi \\ \mathcal{U}_S^{x,1-4} & \rightarrow & \pi(V_d), \end{array}$$

so that ψ is a finite étale map. From what we have shown above, it follows that $e^{-1}(D_1 \cup D_2 \cup G_0 \cup \dots \cup G_m)$ is closed in $\mathcal{V}_S^{x,1-4}$, whose complement is $\widetilde{\mathcal{U}_{m,n}^{x,S}}$. Since ψ is an open map, the image $\mathcal{U}_{m,n}^{x,S} = \psi(\widetilde{\mathcal{U}_{m,n}^{x,S}})$ is open in $\mathcal{U}_S^{x,1-4}$, hence in $\text{Gr}_x(N-1, \mathbb{P}_k^N)$. \square

Lemma 9.1.2. *Suppose $r = 1$ and $1 \leq n \leq d-2$. If $\mathcal{U}_{n,n}^{x,S} \neq \emptyset$, then $\mathcal{U}_{0,n+1}^{x,S} \neq \emptyset$.*

Proof. Set $T = S \cup (f(A_Z) \setminus \{x\})$, where A_Z is as in Claim 1 of Lemma 9.1.1. This T is a finite subset of Z of closed points. Apply Lemma 8.2.5 with S replaced by T . We conclude that $\mathcal{U}_T^{x,1-4}$ is a dense open subset of $\text{Gr}_x(N-1, \mathbb{P}_k^N)$. In particular, $\mathcal{U}_{0,1}^{x,S}$ is a dense open subset of $\text{Gr}_x(N-1, \mathbb{P}_k^N)$.

If $\mathcal{U}_{n,n}^{x,S} \neq \emptyset$, then by Lemma 9.1.1, it is a dense open subset of $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$. In particular, $\mathcal{U}_{n,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$ is dense open in $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$. But, since $\mathcal{U}_{n,n}^{x,S} \cap \mathcal{U}_T^{x,1-4} \subset \mathcal{U}_{0,n+1}^{x,S}$, we have $\mathcal{U}_{0,n+1}^{x,S} \neq \emptyset$ as desired. \square

Lemma 9.1.3. *Suppose $r = 1$, $1 \leq n \leq d-1$, and $0 \leq m \leq n-1$. If $\mathcal{U}_{m,n}^{x,S} \neq \emptyset$, then $\mathcal{U}_{m+1,n}^{x,S} \neq \emptyset$.*

Proof. Let $T \subset X \setminus \{x\}$ be as in Lemma 9.1.2. If $\mathcal{U}_{m,n}^{x,S} \neq \emptyset$, then by Lemma 9.1.1, it is a dense open subset of $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$. In particular, $\mathcal{U}_{m,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$ is dense open in $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$. Fix an element H_0 of $\mathcal{U}_{m,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$ with $H_0 \cap X = \{x = x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_d\}$.

Since $N \gg 0$, there is a one-parameter family (isomorphic to \mathbb{P}_k^1) \mathcal{B} in $\mathrm{Gr}_x(N-1, \mathbb{P}_k^N)$ containing H_0 such that every member of this family passes through $\{x_0, x_{m+1}\}$ and a general member does not pass through x_{n+1} . Since $H_0 \in \mathcal{U}_{m,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$, a general member of \mathcal{B} is in $\mathcal{U}_{m,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$. Let $W \subset \mathcal{B} \cap \mathcal{U}_{m,n}^{x,S} \cap \mathcal{U}_T^{x,1-4}$ be a smooth affine irreducible (rational) curve containing H_0 .

Let V_d be as in (9.1) and let $W \rightarrow \pi(V_d)$ be the map $H \mapsto \sum_{i=1}^d [y_i]$, where $H \cap X = \{x = x_0, y_1, \dots, y_d\}$, and $\pi : X^d \rightarrow \mathrm{Sym}^d(X)$ is as in the paragraph above (9.6). This yields a Cartesian square

$$(9.7) \quad \begin{array}{ccc} W' & \xrightarrow{e} & V_d \\ \psi \downarrow & & \downarrow \pi \\ W & \rightarrow & \pi(V_d), \end{array}$$

so that ψ is finite and étale. Observe that W' is irreducible.

Let $D_1, D_2 \subset V_d$ be as given by (9.2), and for $0 \leq i \leq m+1$, let $G_i \subset V_d$ be given by (9.3). We saw in Claim 2 of Lemma 9.1.1 that D_1, D_2, G_i are closed in V_d .

Claim : $\mathcal{Y} := e^{-1}(D_1 \cup D_2 \cup G_0 \cup \dots \cup G_{m+1})$ is finite.

(\cdot) By the definition of W , we have $e^{-1}(D_2) = \emptyset$.

Since $H_0 \notin e^{-1}(D_1)$ and $H_0 \notin e^{-1}(G_i)$ for $1 \leq i \leq m$, we see that $e^{-1}(D_1)$ and $e^{-1}(G_i)$, $1 \leq i \leq m$, are finite, being proper closed subsets of an irreducible curve W' . By our choice of W , no member of W passes through $f(A_Z) \setminus \{x\}$, hence $e^{-1}(G_0) = \emptyset$.

To show that $e^{-1}(G_{m+1})$ is finite, consider the composite map $q : W' \xrightarrow{e} V_d \rightarrow X^2$ which takes $\underline{H} = (H, y_1, \dots, y_d)$ to $(y_{m+1}, y_{n+1}) \in X^2$. Since each $H \in W$ contains x_{m+1} , the composition of q with the first projection $X^2 \rightarrow X$ is the constant map which takes all $\underline{H} \in W'$ to x_{m+1} . On the other hand, since a general member of W does not contain x_{n+1} , we see that the composition of q with the second projection $X^2 \rightarrow X$ is not a constant map. In other words, the map q is not constant. In particular, the image $q(W')$ is an irreducible curve such that $q(W') \subset \{x_{m+1}\} \times X$. Let $W' \xrightarrow{u} q(W') \xrightarrow{v} X$ denote the map $(H, y_1, \dots, y_d) \mapsto y_{n+1}$. Since u and v are both non-constant morphisms of irreducible curves, both u and v are dominant and quasi-finite.

On the other hand, one checks that $e^{-1}(G_{m+1})$ is a subset of $\{(H, y_1, \dots, y_d) \in W' \mid y_{n+1} \in S_1\}$, where $S_1 := f(g^{-1}(g(Z_{x_{m+1}})))$. Since f and g are finite by Claim

1 of Lemma 9.1.1, the set S_1 must be finite. Hence, $(v \circ u)^{-1}(S_1)$ is finite. Since $e^{-1}(G_{m+1}) \subset (v \circ u)^{-1}(S_1)$, one deduces $|e^{-1}(G_{m+1})| < \infty$. This proves the Claim.

Now, choose any $\underline{H} = (H, y_1, \dots, y_d) \in W' \setminus \mathcal{Y}$. It is clear from the choice of W and \mathcal{Y} that $H \in \mathcal{U}_{m+1,n}^{x,S}$. This completes the proof. \square

Proposition 9.1.4. *Suppose $r = 1$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in X . Then for every $1 \leq n \leq d-1$ and $0 \leq m \leq n$, the set $\mathcal{U}_{m,n}^{x,S} \subset \text{Gr}_x(N-1, \mathbb{P}_k^N)$ is a dense open subset. In particular, the set $\mathcal{U}_{\text{adm}}^{x,S} \subset \text{Gr}_x(N-1, \mathbb{P}_k^N)$ consisting of (Z, x) -admissible hyperplanes is a dense open subset.*

Proof. By Lemma 9.1.1, each $\mathcal{U}_{m,n}^{x,S}$ is open. We saw in the proof of Lemma 9.1.2 that $\mathcal{U}_{0,1}^{x,S}$ is open and dense. Applying Lemmas 9.1.2 and 9.1.3 repeatedly, we conclude that each $\mathcal{U}_{m,n}^{x,S}$ is nonempty and open, thus dense open. The last assertion follows because $\mathcal{U}_{\text{adm}}^{x,S} = \mathcal{U}_{d-1,d-1}^{x,S}$. \square

Proposition 9.1.5. *Suppose $r \geq 2$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in X . After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with a Veronese embedding, one has the following:*

For a general $L_0 \in \text{Gr}(N-r+1, \mathbb{P}_k^N)$ and for $1 \leq n \leq d-1$, $0 \leq m \leq n$, the set $\mathcal{U}_{m,n}^{x,S,L_0} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ is a dense open subset. In particular, the set $\mathcal{U}_{\text{adm}}^{x,S,L_0} \subset \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$, consisting of hyperplanes which are (Z, x) -admissible relative to L_0 , is a dense open subset.

Proof. The last assertion follows from the first by taking $m = n = d-1$, so we prove the first one. Choose a reembedding $\eta : X \hookrightarrow \mathbb{P}_k^N$, a general $L_0 \in \text{Gr}(N-r+1, \mathbb{P}_k^N)$ and $C = L_0 \cap X$ as in Lemma 8.3.4. Set $S' = (C \setminus \{x\}) \cap (f(Z_{\text{sing}}) \cup X_{\text{nfs}} \cup S)$ and $W = f^{-1}(C)$. We use the notations from the proof of Lemma 8.3.4. Recall the smooth surjective morphism $\theta_{L_0} : \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N) \rightarrow \text{Gr}_x(N-r, L_0)$ from (8.3). By applying Proposition 9.1.4 to C and $S' \subset C \setminus \{x\}$, we know that the subset $\mathcal{U}_{m,n}^{x,S'} \subset \text{Gr}_x(N-r, L_0)$ is dense open. Hence, its inverse image via θ_{L_0} is a dense open subset of $\text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$. Thus, it only remains to show that $\mathcal{U}_{m,n}^{x,S,L_0} = \theta_{L_0}^{-1}(\mathcal{U}_{m,n}^{x,S'})$. We have already shown in the proof of Lemma 8.3.4 that $\theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,1-4}) = \mathcal{U}_{S',L_0,1-4}^{x,S}$.

Since $W = f^{-1}(C)$, we see that $Z_y = W_y$ and hence $g(Z_y) = g(W_y)$ for any $y \in C$. It follows that for any $H \in \text{Gr}_x^{\text{tr}}(L_0, N-1, \mathbb{P}_k^N)$ with $\theta_{L_0}(H) \cap X = (H \cap L_0) \cap C = \{x = x_0, x_1, \dots, x_d\}$, the conditions (5)~(6) of Definition 8.3.2 are satisfied if and only if the conditions (5)~(6) of Definition 8.2.2 are satisfied for $\theta_{L_0}(H)$ with X replaced by C . In other words, $\theta_{L_0}^{-1}(\mathcal{U}_{S'}^{x,5-6}) = \mathcal{U}_{S',L_0,5-6}^{x,S}$. We conclude that $\mathcal{U}_{m,n}^{x,S,L_0} = \theta_{L_0}^{-1}(\mathcal{U}_{m,n}^{x,S'})$. This proves the proposition. \square

9.2. Openness of the admissible loci II. In §9.2, we extend the openness assertion on the admissible loci to the situation where we replace a fixed smooth closed point $x \in X$ by any finite set of smooth closed points of X .

Let $\eta : X \hookrightarrow \mathbb{P}_k^N$ be a closed embedding such that $N \gg r = \dim(X)$ and of degree $d+1 \gg 0$ as in § 8.2. Recall from §7.2 that for closed subsets $Y, Y' \subset X$ with $Y \cap Y' = \emptyset$, the set $\text{Gr}_Y(Y', n, \mathbb{P}_k^N)$ denotes the set of n -dimensional linear subspaces of \mathbb{P}_k^N which contain Y but do not intersect Y' . Recall for a closed point

$x \in X$ and $L \in \text{Gr}(x, n, \mathbb{P}_k^N)$, the set $C_x(L)$ is the linear span of x and L in \mathbb{P}_k^N . This is the unique element of $\text{Gr}(n+1, \mathbb{P}_k^N)$ containing L and x . The association $L \mapsto C_x(L)$ defines a smooth surjective affine morphism of schemes

$$(9.8) \quad \vartheta_x : \text{Gr}(x, n, \mathbb{P}_k^N) \rightarrow \text{Gr}_x(n+1, \mathbb{P}_k^N); \quad \vartheta_x(L) = C_x(L)$$

of relative dimension $n+1$ whose fiber over a point M is $\text{Gr}(x, n, M)$. In fact, it is a vector bundle morphism of rank $n+1$. If L_0 is any proper linear subspace of \mathbb{P}_k^N containing x , then ϑ_x induces a smooth surjective map

$$(9.9) \quad \vartheta_x^{L_0} : \text{Gr}^{\text{tr}}(x, L_0, n, \mathbb{P}_k^N) \rightarrow \text{Gr}_x^{\text{tr}}(L_0, n+1, \mathbb{P}_k^N),$$

where recall that $\text{Gr}^{\text{tr}}(S, L_0, n, \mathbb{P}_k^N) = \text{Gr}^{\text{tr}}(L_0, n, \mathbb{P}_k^N) \cap \text{Gr}(S, n, \mathbb{P}_k^N)$.

Lemma 9.2.1. *Given a proper linear subspace $L_0 \subset \mathbb{P}_k^N$ and an element $L \in \text{Gr}^{\text{tr}}(L_0, N-r, \mathbb{P}_k^N)$ intersecting X properly, the set $\text{Gr}^{\text{tr}}(X, L_0, N-r-1, L) = \text{Gr}(X, N-r-1, L) \cap \text{Gr}^{\text{tr}}(L_0, N-r-1, L)$ is a dense open of $\text{Gr}(N-r-1, L)$.*

Proof. Since L intersects X properly in \mathbb{P}^N and $\text{codim}_{\mathbb{P}^N}(L) = \dim(X) = r$, we see that $D := X \cap L$ is a 0-dimensional closed subscheme of L . In particular, $|D| < \infty$. Since $N \gg r$, the subscheme $G(|D|) := \{M \in \text{Gr}(N-r-1, L) \mid M \cap |D| \neq \emptyset\}$ is a proper closed subset of $\text{Gr}(N-r-1, L)$. Hence, $\text{Gr}(X, N-r-1, L) = \text{Gr}(N-r-1, L) \setminus G(|D|)$ is dense open in $\text{Gr}(N-r-1, L)$.

Since the intersection $L \cap L_0$ is transversal, $\text{Gr}^{\text{tr}}(L_0, N-r-1, L)$ is dense open in $\text{Gr}(N-r-1, L)$, thus $\text{Gr}^{\text{tr}}(X, L_0, N-r-1, L)$ is dense open in $\text{Gr}(N-r-1, L)$, as desired. \square

Definition 9.2.2. Let $x \in X_{\text{fs}}$ be a closed point and let $S \subset X \setminus \{x\}$ be a finite set of closed points. For $r = 1$, let $\mathcal{U}_{\text{adm}}^{x, S, N-2}(Z) \subset \text{Gr}(N-2, \mathbb{P}_k^N)$ be the set of linear subspaces M such that

- (1) $M \cap X = \emptyset$ and
- (2) the linear space $C_x(M)$ is (Z, x) -admissible (see Definition 8.2.2).

For $r \geq 2$ and $L \in \text{Gr}(N-r+1, \mathbb{P}_k^N)$, let $\mathcal{U}_{\text{adm}}^{x, S, L, N-2}(Z) \subset \text{Gr}(N-2, \mathbb{P}_k^N)$ be the set of all $(N-2)$ -dimensional linear subspaces M such that

- (1) M intersects L transversely,
- (2) $M \cap L \cap X = \emptyset$, and
- (3) the linear space $C_x(M)$ is (Z, x) -admissible relative to L (see Definition 8.3.2).

Definition 9.2.3. Let $H_0 \hookrightarrow \mathbb{P}_k^N$ be a hyperplane disjoint from $S \cup \{x\}$. For $r = 1$, let $\mathcal{U}_{\text{adm}}^{x, S, N-2}(Z, H_0)$ denote the intersection $\mathcal{U}_{\text{adm}}^{x, S, N-2}(Z) \cap \text{Gr}(N-2, H_0)$. For $r \geq 2$, let $\mathcal{U}_{\text{adm}}^{x, S, L, N-2}(Z, H_0)$ denote the intersection $\mathcal{U}_{\text{adm}}^{x, S, L, N-2}(Z) \cap \text{Gr}(N-2, H_0)$.

Lemma 9.2.4. *Let $x \in X_{\text{fs}}$ be a closed point and let $S \subset X \setminus \{x\}$ be a finite set of closed points.*

- (A) *Suppose $r = 1$. Then, given a hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from $S \cup \{x\}$, the set $\mathcal{U}_{\text{adm}}^{x, S, N-2}(Z, H_0)$ is dense open in $\text{Gr}(N-2, H_0)$.*
- (B) *Suppose $r \geq 2$. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with a Veronese embedding if necessary, one has the following:*

given a hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from $S \cup \{x\}$ and a general $L_0 \in \text{Gr}^{\text{tr}}(H_0, N - r + 1, \mathbb{P}_k^N)$, the set $\mathcal{U}_{\text{adm}}^{x,S,L_0,N-2}(Z, H_0)$ is dense open in $\text{Gr}(N - 2, H_0)$.

Proof. We often drop Z from the notations when we write $\mathcal{U}_{\text{adm}}^{x,S,N-2}(Z), \mathcal{U}_{\text{adm}}^{x,S,N-2}(Z, H_0), \mathcal{U}_{\text{adm}}^{x,S,L_0,N-2}(Z)$, and $\mathcal{U}_{\text{adm}}^{x,S,L_0,N-2}(Z, H_0)$.

Suppose $r = 1$. It follows from Proposition 9.1.4 that $\mathcal{U}_{\text{adm}}^{x,S}$ is a dense open subset of $\text{Gr}_x(N - 1, \mathbb{P}_k^N)$. We have seen previously that the map $\vartheta_x : \text{Gr}(x, N - 2, \mathbb{P}_k^N) \rightarrow \text{Gr}_x(N - 1, \mathbb{P}_k^N)$ of (9.8) is a vector bundle of rank $N - 1$. One checks that $\text{Gr}(N - 2, H_0) \hookrightarrow \text{Gr}(x, N - 2, \mathbb{P}_k^N)$ is a closed immersion and the restriction $\vartheta_{x,H_0} : \text{Gr}(N - 2, H_0) \rightarrow \text{Gr}_x(N - 1, \mathbb{P}_k^N)$ is an isomorphism. It follows that $\vartheta_{x,H_0}^{-1}(\mathcal{U}_{\text{adm}}^{x,S})$ is a dense open subset of $\text{Gr}(N - 2, H_0)$. Since $\text{Gr}(N - 2, H_0)$ is irreducible, we conclude from Lemma 7.2.3 that $\mathcal{U}_{\text{adm}}^{x,S,N-2}(H_0) = \text{Gr}(X, N - 2, H_0) \cap \vartheta_{x,H_0}^{-1}(\mathcal{U}_{\text{adm}}^{x,S})$ is a dense open subset of $\text{Gr}(N - 2, H_0)$. This proves (A).

Suppose now that $r \geq 2$. Choose a reembedding $\eta : X \hookrightarrow \mathbb{P}_k^N$, a general $L_0 \in \text{Gr}^{\text{tr}}(H_0, N - r + 1, \mathbb{P}_k^N)$ and $C = L \cap X$ as in Lemma 8.3.4.

It follows from Proposition 9.1.5 that $\mathcal{U}_{\text{adm}}^{x,S,L_0}$ is a dense open subset of $\text{Gr}_x^{\text{tr}}(L_0, N - 1, \mathbb{P}_k^N)$. We know the map $\vartheta_x^{L_0} : \text{Gr}^{\text{tr}}(x, L_0, N - 2, \mathbb{P}_k^N) \rightarrow \text{Gr}_x^{\text{tr}}(L_0, N - 1, \mathbb{P}_k^N)$ of (9.9) is smooth and surjective. Consider its restriction

$$(9.10) \quad \vartheta_{x,H_0} : \text{Gr}^{\text{tr}}(L_0, N - 2, H_0) \rightarrow \text{Gr}_x^{\text{tr}}(L_0, N - 1, \mathbb{P}_k^N).$$

One checks that this map is an inclusion whose image is the dense open subset $\text{Gr}_x^{\text{tr}}(L_0 \cap H_0, N - 1, \mathbb{P}_k^N)$. On the other hand, $H_0 \cap \{x\} = \emptyset$ implies that $\text{Gr}_x^{\text{tr}}(L_0 \cap H_0, N - 1, \mathbb{P}_k^N) = \text{Gr}_x^{\text{tr}}(L_0, N - 1, \mathbb{P}_k^N)$. In particular, (9.10) is an isomorphism and we conclude that $\vartheta_{x,H_0}^{-1}(\mathcal{U}_{\text{adm}}^{x,S,L_0})$ is dense open in $\text{Gr}^{\text{tr}}(L_0, N - 2, H_0)$ and hence dense open in $\text{Gr}(N - 2, H_0)$. Combining this with Lemma 7.2.3, we conclude that $\mathcal{U}_{\text{adm}}^{x,S,L_0,N-2}(H_0) = \text{Gr}(X, N - 2, H_0) \cap \vartheta_{x,H_0}^{-1}(\mathcal{U}_{\text{adm}}^{x,S,L_0})$ is dense open in $\text{Gr}(N - 2, H_0)$. This proves (B). \square

Definition 9.2.5. Let $S \subset X_{\text{fs}}$ be a finite set of distinct closed points and let $H_0 \hookrightarrow \mathbb{P}_k^N$ be a hyperplane disjoint from S . For $r = 1$, let $\mathcal{U}_{\text{adm}}^{S,N-2}(Z, H_0)$ be the subset of $\text{Gr}(N - 2, H_0)$ consisting of linear subspaces $L \in \text{Gr}(N - 2, H_0)$ such that $L \in \mathcal{U}_{\text{adm}}^{x,S \setminus \{x\},N-2}(Z, H_0)$ for each $x \in S$.

For $r \geq 2$ and $L_0 \in \text{Gr}^{\text{tr}}(H_0, N - r + 1, \mathbb{P}_k^N)$, let $\mathcal{U}_{\text{adm}}^{S,L_0,N-2}(Z, H_0)$ be the subset of $\text{Gr}(N - 2, H_0)$ consisting of linear subspaces $L \in \text{Gr}(N - 2, H_0)$ such that $L \in \mathcal{U}_{\text{adm}}^{x,S \setminus \{x\},L_0,N-2}(Z, H_0)$ for each $x \in S$.

Theorem 9.2.6. Let $S = \{x_1, \dots, x_n\} \subset X_{\text{fs}}$ be a finite set of distinct closed points.

- (A) Suppose $r = 1$. Then given a hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from S , the set $\mathcal{U}_{\text{adm}}^{S,N-2}(Z, H_0)$ is dense open in $\text{Gr}(N - 2, H_0)$.
- (B) Suppose $r \geq 2$. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with a Veronese embedding if necessary, one has the following: given a hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from S and a general $L_0 \in \text{Gr}^{\text{tr}}(H_0, N - r + 1, \mathbb{P}_k^N)$, the set $\mathcal{U}_{\text{adm}}^{S,L_0,N-2}(Z, H_0)$ is dense open in $\text{Gr}(N - 2, H_0)$.

Proof. When $r = 1$, it follows directly from Lemma 9.2.4 that the set $\mathcal{U}_{\text{adm}}^{S, N-2}(Z, H_0) = \bigcap_{i=1}^n \mathcal{U}_{\text{adm}}^{x_i, S \setminus \{x_i\}, N-2}(Z, H_0)$ is a dense open subset of $\text{Gr}(N-2, H_0)$.

When $r \geq 2$, it follows from Lemma 9.2.4 that after replacing the embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ by its composition with a Veronese embedding, the following holds: for a general $L_0 \in \text{Gr}^{\text{tr}}(H_0, N-r+1, \mathbb{P}_k^N)$, $\mathcal{U}_{\text{adm}}^{x_i, S \setminus \{x_i\}, L_0, N-2}(Z, H_0)$ is a dense open subset of $\text{Gr}(N-2, H_0)$ for each $1 \leq i \leq n$. In particular, $\mathcal{U}_{\text{adm}}^{S, L_0, N-2}(Z, H_0) = \bigcap_{i=1}^n \mathcal{U}_{\text{adm}}^{x_i, S \setminus \{x_i\}, L_0, N-2}(Z, H_0)$ is also a dense open subset of $\text{Gr}(N-2, H_0)$. \square

10. THE sfs-MOVING LEMMA III: ADMISSIBLE LINEAR PROJECTIONS

Now assume k is any infinite perfect field and let \bar{k} be its algebraic closure. For $X \in \mathbf{Sch}_k^{\text{ess}}$, let $\pi_X : X_{\bar{k}} \rightarrow X$ be the base change to \bar{k} .

10.1. Admissible finite subsets. Let X be an irreducible quasi-projective scheme of dimension $r \geq 1$ with a smooth dense open subset X_{fs} . Let $x \in X_{\text{fs}}$ be a closed point. Let B be a geometrically integral smooth affine k -scheme of positive dimension, and let \widehat{B} be a geometrically integral smooth compactification of B . Given a closed subscheme $Z \subset X \times \widehat{B}$, let $f : Z \rightarrow X$ and $g : Z \rightarrow \widehat{B}$ be the projection maps.

Definition 10.1.1. Let $Z \subset X \times \widehat{B}$ be an irreducible fs-cycle in the sense of Definition 8.1.1. A finite subset $D_x \subset X$ of distinct closed points is called (Z, x) -admissible, if the following hold:

- (1) $D_x \subset X_{\text{fs}}$.
- (2) $x \in D_x$.
- (3) Z is regular at the points of $(D_x \setminus \{x\}) \times \widehat{B}$.
- (4) Either $Z = X \times \{b\}$ for some closed point $b \in \widehat{B}$, or $g(Z_{x_i}) \cap g(Z_{x_j}) = \emptyset$ for each distinct pair $x_i \neq x_j$ in D_x .

For an fs-cycle Z on $X \times \widehat{B}$ in the sense of Definition 8.1.1 with irreducible components $\{Z_1, \dots, Z_s\}$, we say that D_x is (Z, x) -admissible if it is (Z_i, x) -admissible for each $1 \leq i \leq s$.

The following immediately follows from the definition and smoothness of X_{fs} .

Lemma 10.1.2. Let Z be an fs-cycle in the sense of Definition 8.1.1. Write $Z = Z_1 + Z_2$, where each component of Z_1 is rectifiable while each component of Z_2 is not. Then a finite set $D_x = \{x = x_0, x_1, \dots, x_d\}$ of distinct closed points on X is (Z, x) -admissible if and only if it is (Z_1, x) -admissible.

10.2. Global nature of admissible linear projections. For an embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ of a projective k -scheme X into a big enough projective space, we prove some results about the maps $\phi : X \rightarrow \mathbb{P}_k^r$ obtained from the linear projections away from “admissible” $(N-r-1)$ -dimensional linear subspaces in \mathbb{P}_k^N .

Proposition 10.2.1. Let X be a projective k -scheme of dimension $r \geq 1$ and let Z be an fs-cycle on $X \times \widehat{B}$ as in Definition 8.1.1, with irreducible components $\{Z_1, \dots, Z_s\}$.

Let $\Sigma = \{x_1, \dots, x_n\} \subset X_{\text{fs}}$ be a finite subset of distinct closed points and let $Y \subset X$ be a closed subscheme of dimension $\leq r-1$. Then there is a closed

embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ such that for a given hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from Σ and a general linear subspace $L \in \text{Gr}(N - r - 1, H_0)$, the linear projection $\phi_L : \mathbb{P}_k^N \setminus L \rightarrow \mathbb{P}_k^r$ restricts to a finite map $\phi : X \rightarrow \mathbb{P}_k^r$ with the following additional properties:

- (1) ϕ is étale at $\phi^{-1}(\phi(\Sigma))$.
- (2) $\phi(x_i) \neq \phi(x_j)$ for $1 \leq i \neq j \leq n$.
- (3) $k(\phi(x)) \xrightarrow{\sim} k(x)$ for each $x \in \phi^{-1}(\phi(\Sigma))$.
- (4) $\phi^{-1}(\phi(x_i))$ is (Z, x_i) -admissible for each $1 \leq i \leq n$.
- (5) $L^+(\Sigma) \cap Y = \emptyset$.

Proof. By Lemma 10.1.2, we may assume Z is rectifiable. For $1 \leq i \leq n$, let $S_i = \pi_X^{-1}(x_i) = \{x_i^1, \dots, x_i^{r_i}\}$ and set $S = \bigcup_{i=1}^n S_i \subset X_{\text{fs}, \bar{k}}$. In particular, S is a finite subset of smooth closed points of $X_{\bar{k}}$.

We choose a big enough closed embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ such that the inclusion $X_{\bar{k}} \hookrightarrow \mathbb{P}_{\bar{k}}^N$ satisfies the assertions of Theorem 9.2.6 with the above $S \subset X_{\bar{k}}$ so that $S \cap H_{0, \bar{k}} = \emptyset$. Recall for a closed subscheme $W \subset \mathbb{P}_{\bar{k}}^N$, the set $C_S(W)$ is the union of the secant lines joining the points of S and W . One knows that $\dim(C_S(W)) \leq \dim(W) + 1$.

If $r = 1$, we take any $L \in \mathcal{U}_{\text{adm}}^{S, N-2}(Z_{\bar{k}}, H_{0, \bar{k}}) \cap \text{Gr}(C_S(Y_{\bar{k}}), N-2, H_{0, \bar{k}})$. It follows from Lemma 7.2.3 that $\text{Gr}(C_S(Y_{\bar{k}}), N-2, H_{0, \bar{k}})$ is a dense open subset of $\text{Gr}(N-2, H_{0, \bar{k}})$.

If $r \geq 2$, we choose a general $L_0 \in \text{Gr}^{\text{tr}}(H_{0, \bar{k}}, N-r+1, \mathbb{P}_{\bar{k}}^N)$ such that $L_0 \cap X_{\bar{k}}$ is a curve C , none of whose component is contained in $Y_{\bar{k}}$. Recall $\text{Gr}^{\text{tr}}(C_S(Y_{\bar{k}} \cap C), L_0, N-2, H_{0, \bar{k}})$ is the set of linear subspaces in $\text{Gr}(N-2, H_{0, \bar{k}})$ which intersect L_0 transversely and do not intersect $C_S(Y_{\bar{k}} \cap C)$. By Lemmas 7.2.3 and 7.2.4, it is a dense open subset of $\text{Gr}(N-2, H_{0, \bar{k}})$. We take $L = M \cap L_0$ for any $M \in \mathcal{U}_{\text{adm}}^{S, L_0, N-2}(Z_{\bar{k}}, H_{0, \bar{k}}) \cap \text{Gr}^{\text{tr}}(C_S(Y_{\bar{k}} \cap C), L, N-2, H_{0, \bar{k}})$.

As shown in the Step 2 of the proof of Lemma 7.3.2, we can find dense subsets of $\mathcal{U}_{\text{adm}}^{S, N-2}(Z_{\bar{k}}, H_{0, \bar{k}}) \cap \text{Gr}(C_S(Y_{\bar{k}}), N-2, H_{0, \bar{k}})$, $\text{Gr}(N-r+1, \mathbb{P}_{\bar{k}}^N)$ and $\mathcal{U}_{\text{adm}}^{S, L_0, N-2}(Z_{\bar{k}}, H_{0, \bar{k}}) \cap \text{Gr}^{\text{tr}}(C_S(Y_{\bar{k}} \cap C), L_0, N-2, H_{0, \bar{k}})$, all of whose points L, L_0 , and M , respectively, are defined over k . Since $X \cap L = \emptyset$, we get a finite map $\phi = \phi_L : X \rightarrow \mathbb{P}_k^r$ over k . We now show that such ϕ satisfies the properties (1)~(5) of the proposition.

It follows from the admissibility condition for $C_x(L)$ for all $x \in S$ (see Definition 8.3.2) that $\phi_{\bar{k}}^{-1}(\phi_{\bar{k}}(S)) \subset X_{\text{fs}, \bar{k}}$ and equivalently, $\phi^{-1}(\phi(\Sigma)) \subset X_{\text{fs}}$. It follows from the admissibility of $C_x(L)$ and Lemma 10.2.2 below that $\phi_{\bar{k}}$ is étale at all points of $\phi_{\bar{k}}^{-1}(\phi_{\bar{k}}(S))$. But this is equivalent (by descent) to saying that ϕ is étale at all points of $\phi^{-1}(\phi(\Sigma))$. This proves (1).

Since the map $\phi_{\bar{k}}$ is injective on S , the properties (2) and (3) follow directly from the Claim in the proof of Lemma 7.3.2.

For $1 \leq i \leq n$, set $\Sigma_i = \phi^{-1}(\phi(x_i))$. We have already shown that $\Sigma_i \subset X_{\text{fs}}$. It follows from the admissibility condition that $Z_{\bar{k}}$ is smooth at all points of $\pi_X^{-1}(\Sigma_i \setminus \{x_i\}) \times \widehat{B}_{\bar{k}}$. We conclude from the faithfully flat descent of smoothness that Z is smooth at all points of $(\Sigma_i \setminus \{x_i\}) \times \widehat{B}$. The third and the fourth conditions of Definition 10.1.1 follow directly from the Definition 9.2.5 of $\mathcal{U}_{\text{adm}}^{S, N-2}$ and $\mathcal{U}_{\text{adm}}^{S, L_0, N-2}$. This shows that each Σ_i is (Z, x_i) -admissible, proving (4). The

property (5) follows at once from our choice of L , as shown in the proof of the last property of Lemma 7.3.2. \square

In the middle of the proof of above Proposition 10.2.1, we used the following:

Lemma 10.2.2. *Let $X \hookrightarrow \mathbb{P}_k^N$ be a closed embedding of a projective k -scheme of dimension $r \geq 1$. Let $L \subset \mathbb{P}_k^N$ be a linear subspace of dimension $N - r - 1$ such that $X \cap L = \emptyset$. Regard \mathbb{P}_k^r as a linear subspace of \mathbb{P}_k^N of dimension r , not intersecting L , and let $\mathfrak{p} \in \mathbb{P}_k^r$ be a closed point such that $C_{\mathfrak{p}}(L) \cap X_{\text{sing}} = \emptyset$. Then the map $\phi_L : X \rightarrow \mathbb{P}_k^r$ obtained by the linear projection away from L , is finite and étale over an affine neighborhood of \mathfrak{p} in \mathbb{P}_k^r if and only if $C_{\mathfrak{p}}(L)$ intersects X transversely in \mathbb{P}_k^N .*

Proof. Suppose that $C_{\mathfrak{p}}(L)$ intersects X transversely in \mathbb{P}_k^N and let E be this scheme-theoretic intersection, with $\text{Supp}(E) = \{x_0, \dots, x_r\}$. Since k is perfect, the transversal intersection is equivalent to saying that E is smooth (but disconnected). However, as $L \subset C_{\mathfrak{p}}(L)$ and $X \cap L = \emptyset$, we see that $C_{\mathfrak{p}}(L) \cap X = (C_{\mathfrak{p}}(L) \setminus L) \cap X$ as schemes. Observe that the latter is same as the scheme-theoretic fiber $\phi_L^*(\mathfrak{p})$. In other words, when $U \subset \mathbb{P}_k^r$ is an affine open neighborhood of P , the diagram

$$(10.1) \quad \begin{array}{ccc} E & \longrightarrow & \phi_L^{-1}(U) \\ \phi_L^{\mathfrak{p}} \downarrow & & \downarrow \phi_L \\ \text{Spec}(k(\mathfrak{p})) & \longrightarrow & U \end{array}$$

is Cartesian such that $\phi_L^{\mathfrak{p}}$ is smooth. Since ϕ_L is a finite map of affine schemes over k , by [36, Theorem 24.3], it is flat over an affine neighborhood of \mathfrak{p} . We can now apply [16, Exercise III-10.2] to conclude that there is an affine neighborhood of \mathfrak{p} in U over which the map ϕ_L is smooth, thus finite and étale.

The converse is easy to see, because smoothness of ϕ_L over a neighborhood of \mathfrak{p} implies that the map $\phi_L^{\mathfrak{p}}$ in (10.1) is étale. Since $k \hookrightarrow k(\mathfrak{p})$ is smooth, this precisely means that $E = C_{\mathfrak{p}}(L) \cap X$ is smooth, which in turn is equivalent to saying that this intersection is transverse. \square

10.3. Local nature of admissible linear projections. Now let X be a connected smooth affine k -scheme of dimension $r \geq 1$ and let $\Sigma \subset X$ be a finite set of closed points. Let $X \hookrightarrow \mathbb{A}_k^m$ be a closed embedding and let \overline{X} denote the closure of X under the inclusion $\mathbb{A}_k^m \hookrightarrow \mathbb{P}_k^m$. Let $H_{m,0}$ denote the hyperplane $\mathbb{P}_k^m \setminus \mathbb{A}_k^m$ as in §7.2.1. Let B be a geometrically integral smooth affine k -scheme of positive dimension, and let \widehat{B} be a geometrically integral smooth compactification of B . Let Z be an fs-cycle on $X \times \widehat{B}$ in the sense of Definition 8.1.1, with irreducible components $\{Z_1, \dots, Z_s\}$. Let \overline{Z}_i and \overline{Z} be the closures of Z_i and Z in $\overline{X} \times \widehat{B}$. Notice that \overline{Z} is an fs-cycle, too. It is a rectifiable cycle if Z is so. Let $Y \subset \overline{X}$ be a closed subscheme of dimension at most $r - 1$.

Lemma 10.3.1. *Given a cycle Z on $X \times \widehat{B}$ as above, there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$ such that the following hold:*

(1) *The linear projection away from L defines a Cartesian square*

$$(10.2) \quad \begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{A}_k^r & \hookrightarrow & \mathbb{P}_k^r \end{array}$$

such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subset \mathbb{A}_k^r$.

- (2) *The map $\phi : X \rightarrow \mathbb{A}_k^r$ is étale over an affine neighborhood U of $\phi(\Sigma)$.*
- (3) *$\phi(x) \neq \phi(x')$ for $x \neq x' \in \Sigma$.*
- (4) *$k(\phi(x)) \xrightarrow{\sim} k(x)$ for each $x \in \phi^{-1}(\phi(\Sigma))$.*
- (5) *$L^+(\Sigma) \cap Y = \emptyset$.*
- (6) *$\phi^{-1}(\phi(x))$ is (Z, x) -admissible as in Definition 10.1.1 for each $x \in \Sigma$. In particular, $\phi^{-1}(\phi(\Sigma)) \subset X$.*

Proof. We apply Proposition 10.2.1 to \overline{X} with $(\overline{X})_{\text{fs}} := X$ and $H_0 = H_{N,0}$ (see (7.2)) to get a closed embedding $\eta : \overline{X} \hookrightarrow \mathbb{P}_k^N$ and a linear subspace $L \in \text{Gr}(\overline{X}, N - r - 1, H_{N,0})$ such that the linear projection $\phi_L : \mathbb{P}_k^N \setminus L \rightarrow \mathbb{P}_k^r$ restricts to a finite map $\phi : \overline{X} \rightarrow \mathbb{P}_k^r$ satisfying the properties (3)~(6) of the lemma. By Lemma 7.2.2, the property (1) holds and $\phi(\Sigma) \subset \mathbb{A}_k^r$.

Since ϕ is flat along $\phi(\Sigma)$, by [36, Theorem 24.3], there is an affine open neighborhood $U' \subset \mathbb{A}_k^r$ of $\phi(\Sigma)$ such that the map $\phi^{-1}(U') \rightarrow U'$ is finite and flat. We can now apply [16, Exercise III-10.2] to conclude that there is an affine open neighborhood $U \subset \mathbb{A}_k^r$ of $\phi(\Sigma)$ such that the map $\phi^{-1}(U) \rightarrow U$ is finite and étale. This proves property (2) of the lemma. \square

We have the following improvement of Lemma 10.3.1 when \widehat{B} is of more specific form. Let $n \geq 1$ be an integer. Let A_0, A_1, \dots, A_{n-1} be smooth affine geometrically integral k -schemes of positive dimension, and let $\widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_{n-1}$ be smooth projective geometrically integral k -schemes such that $A_j \subset \widehat{A}_j$ is open for $0 \leq j \leq n-1$. For $1 \leq j \leq n$, we set

$$C_j := \prod_{i=0}^{j-1} A_i, \quad \widehat{C}_j := \prod_{i=0}^{j-1} \widehat{A}_i, \quad B := C_n, \quad \widehat{B} = \widehat{C}_n, \quad \pi_j : \widehat{B} \rightarrow \widehat{C}_j,$$

where π_j is the obvious projection.

Let X be a smooth affine k -scheme of dimension $r \geq 1$ and let $\Sigma \subset X$ be a finite subset of closed points. Let Z be an fs-cycle on $X \times \widehat{B}$ as in Definition 8.1.1, with irreducible components $\{Z_1, \dots, Z_s\}$. For $1 \leq j \leq n$, let $Z^{(j)} := \pi_j(Z)$. Then each $Z^{(j)}$ is an fs-cycle on $X \times \widehat{C}_j$. Fix an affine embedding $X \hookrightarrow \mathbb{A}_k^m$ and let \overline{X} be the Zariski closure of X in \mathbb{P}_k^m . Let $Y \subset \overline{X}$ be a closed subscheme of dimension $\leq r-1$.

Proposition 10.3.2. *Under the above notations, there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$ such that the following hold:*

(1) *The linear projection away from L defines a Cartesian square*

$$(10.3) \quad \begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{A}_k^r & \hookrightarrow & \mathbb{P}_k^r \end{array}$$

such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subset \mathbb{A}_k^r$.

- (2) *The map $\phi : X \rightarrow \mathbb{A}_k^r$ is étale over an affine neighborhood U of $\phi(\Sigma)$.*
- (3) *$\phi(x) \neq \phi(x')$ for $x \neq x' \in \Sigma$.*
- (4) *$k(\phi(x)) \xrightarrow{\sim} k(x)$ for each $x \in \phi^{-1}(\phi(\Sigma))$.*
- (5) *$L^+(\Sigma) \cap Y = \emptyset$.*
- (6) *$\phi^{-1}(\phi(x))$ is $(Z^{(j)}, x)$ -admissible in the sense of Definition 10.1.1 for each $x \in \Sigma$ and each $1 \leq j \leq n$. In particular, $\phi^{-1}(\phi(\Sigma)) \subset X$.*

Proof. It follows from Proposition 10.2.1 that there is an embedding $\eta : X \hookrightarrow \overline{X} \hookrightarrow \mathbb{P}_k^N$ such that for a general linear subspace $L \in \text{Gr}(\overline{X}, N - r - 1, H_{N,0})$ (with $H_{N,0}$ as in Lemma 10.3.1), the linear projection $\phi_L : \mathbb{P}_k^N \setminus L \rightarrow \mathbb{P}_k^r$ restricts to a finite map $\phi : \overline{X} \rightarrow \mathbb{P}_k^r$ satisfying the properties (3)~(6) of the proposition for each $\overline{Z}^{(j)}$. Let $\mathcal{U}^{(j)}$ denote the open subset of $\text{Gr}(N - r - 1, H_{N,0})$ satisfying this property and let $\mathcal{U} = \bigcap_{j=1}^n \mathcal{U}^{(j)}$. We see that for a general member L of $\mathcal{U} \cap \text{Gr}(\overline{X}, N - r - 1, H_{N,0}) \subset \text{Gr}(N - r - 1, H_{N,0})$, the associated linear projection ϕ_L satisfies the properties (3)~(6) of the proposition. Now one can repeat the argument of the proof of Lemma 10.3.1 to complete the rest of the proof. \square

11. THE sfs-MOVING LEMMA IV: THE MAIN RESULTS

Let k be an infinite perfect field. In this section, we complete the proof of Theorem 5.2.3. The interested reader can apply the machines of the paper to the Milnor range of higher Chow cycles, to obtain an analogous result.

11.1. Smoothness for rectifiable cycles. Let X be a connected smooth affine k -scheme. Let Σ be a finite set of closed points of X . Let $X \hookrightarrow \mathbb{A}_k^m$ be a closed embedding. Let \overline{X} be the closure of X in \mathbb{P}_k^m and let $H_{m,0} = \mathbb{P}_k^m \setminus \mathbb{A}_k^m$ as in Section 7.2.1. We now apply Proposition 10.3.2 with $C_j = B_j = \mathbb{A}_k^1 \times \square_k^{j-1}$ and $\widehat{C}_j = \widehat{B}_j = \mathbb{P}_k^1 \times \square_k^{j-1}$ for $1 \leq j \leq n$.

Let Z be an fs-cycle on $X \times \widehat{B}_n$ with irreducible components $\{Z_1, \dots, Z_s\}$ such that each $Z_i \rightarrow X$ is finite and surjective. Let \overline{Z}_i and \overline{Z} denote the closures of Z_i and Z in $\overline{X} \times \widehat{B}_n$. Note that \overline{Z} is an fs-cycle on $\overline{X} \times \widehat{B}_n$. Furthermore, it is a rectifiable cycle if Z is so.

Proposition 11.1.1. *Under the above notations, suppose that Z is rectifiable. Then there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$ such that the following hold:*

- (1) *The linear projection away from L defines a Cartesian square*

$$(11.1) \quad \begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{A}_k^r & \hookrightarrow & \mathbb{P}_k^r \end{array}$$

such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subset \mathbb{A}_k^r$.

- (2) *The map $\phi : X \rightarrow \mathbb{A}_k^r$ is étale over an affine neighborhood U of $\phi(\Sigma)$.*
 (3) *Given any $1 \leq i \leq s$ and irreducible component Z' of $L^+(Z_i)$, the scheme $\pi_j(Z')$ is an fs-cycle which is also smooth at all points over $\Sigma \times \widehat{B}_j$ for each $1 \leq j \leq n$.*

Proof. For each $1 \leq i \leq s$, as in §7.3, fix closed points $x_i \in X$, $b_i \in \widehat{B}_n$ such that $\alpha_i = (x_i, b_i) \in Z_i$, with $\alpha_i \notin Z_j$ if $i \neq j$. Since each $Z_i \neq \emptyset$, such closed points exist. Set $D_0 = \Sigma \cup \{x_1, \dots, x_s\}$ and $E^0 = \{b_1, \dots, b_s\}$. Let Y denote the closure of $f(Z \cap (X \times E^0))$ in \overline{X} . Since Z is rectifiable, we see that $Z \not\subset (X \times E^0)$. Since f is projective, $f(Z \cap (X \times E^0))$ is a closed subset of X dimension $\leq r - 1$. Thus, $Y \subset \overline{X}$ is a closed subset of dimension $\leq r - 1$.

Let $\phi : \overline{X} \rightarrow \mathbb{P}_k^r$ be the finite map satisfying the properties (1)~(6) of Proposition 10.3.2 with $D_0 \subset X$ in the place of ' Σ ' there, and $Y \subset \overline{X}$. This gives an affine neighborhood U of $\phi(D_0)$ and implies the properties (1) and (2) of the Proposition 11.1.1. It remains to prove the property (3). Let $\widehat{\phi} = \phi \times \text{Id}_{\widehat{B}_n}$ and $\widehat{\phi}_j = \phi \times \text{Id}_{\widehat{B}_j}$ for $1 \leq j \leq n$.

For each $1 \leq i \leq s$, recall that $L^+([Z_i])$ is the cycle $\phi^* \circ \phi_*([Z_i]) - [Z_i]$ and $L^+(Z_i)$ is the support of $L^+([Z_i])$. For $L^+(\overline{Z}_i)$, the reader should observe that \overline{X} may not be smooth so that $\widehat{\phi}$ may not be flat, thus $\widehat{\phi}^*$ (in the sense of [11, § 1.7]) may not be defined in general. But it is defined for the map $\phi : \phi^{-1}(U) \rightarrow U$, and our interest is only in this open subset. Hence, by $\widehat{\phi}^* \widehat{\phi}_*([Z_i])$, we shall mean it wherever it is defined. Under the assumptions of the Proposition 11.1.1, we have already shown in the proof of Proposition 7.4.1 that Z_i is not a component of $L^+(Z_i)$. This uses the properties (2), (4) and (5) of Proposition 10.3.2 as well as our choice of $\alpha_i \in Z_i$ and Y .

To prove the remaining part of (3), fix an arbitrary $1 \leq i \leq s$ and $x \in \Sigma$. So, we may replace Z by Z_i . Let $y = \phi(x)$.

Claim 1: $\pi_j(L^+(\overline{Z})) = L^+(\overline{Z}^{(j)})$ for each $1 \leq j \leq n$.

(\because) It is clear that $\pi_j(L^+(\overline{Z})) \subset L^+(\overline{Z}^{(j)})$. To prove the reverse inclusion, let $(x, b_0, \dots, b_{j-1}) \in L^+(\overline{Z}^{(j)})$. This means that there exists $z' = (x', b'_0, \dots, b'_{n-1}) \in \overline{Z}$ such that $(\phi(x), b_0, \dots, b_{j-1}) = \phi \circ \pi_j(x', b'_0, \dots, b'_{n-1}) = (\phi(x'), b'_0, \dots, b'_{j-1})$. But, this implies that $\phi(x) = \phi(x')$ and $b_l = b'_l$ for $0 \leq l \leq j - 1$. Setting $z = (x, b'_0, \dots, b'_{n-1})$, we see that $\widehat{\phi}(z) = \widehat{\phi}(z')$ and $\pi_j(z) = (x, b_0, \dots, b_{j-1})$. This proves Claim 1.

For $1 \leq j \leq n$, let $T^{(j)} = \widehat{\phi}_j(\overline{Z}^{(j)})$ and let $\widetilde{Z}^{(j)}$ denote the scheme-theoretic inverse image of $T^{(j)}$. Set $T = T^{(n)}$ and $\widetilde{Z} = \widetilde{Z}^{(n)}$. It follows from Claim 1 that $\widetilde{Z}^{(j)} = \pi_j(\widehat{\phi}_n^{-1}(T)) = \pi_j(\widetilde{Z})$.

We first show that $L^+(\overline{Z})$ is an fs-cycle. Let Z' be a component of $L^+(\overline{Z})$. Since $Z' \rightarrow \overline{X}$ is a projective morphism of schemes of the same dimension, it is enough to show that this map is quasi-finite. Suppose on the contrary that there is a closed point $x \in \overline{X}$ such that $\dim(Z'_x) \geq 1$ and let $E = p_{\widehat{B}_n}(Z'_x)$, where $p_{\widehat{B}_n} : \overline{X} \times \widehat{B}_n \rightarrow \widehat{B}_n$ is the projection. Since Z' is a part of the residual cycle of \overline{Z} , we must have $\dim(\overline{Z} \cap (D_x \times E)) \geq 1$, where $D_x = \phi^{-1}(\phi(x))$. But this contradicts the fact that \overline{Z} is an fs-cycle and ϕ is finite. We have thus shown that $L^+(\overline{Z})$ is an fs-cycle.

We now show the smoothness of the components of $L^+(\overline{Z})$ along $\Sigma \times \widehat{B}_n$. We have seen before that $\overline{Z} \not\subset \overline{X} \times \{b\}$ for any $b \in \widehat{B}_n$. Set $\widehat{B}_0 = \text{Spec}(k)$, $\overline{Z}^0 = \overline{X}$ and let $m \in \{0, 1, \dots, n-1\}$ be the largest integer such that $\overline{Z}^{(m)} = \overline{X} \times \{b\}$ for some $b \in \widehat{B}_m$. It follows from Claim 1 that $(L^+(\overline{Z}))^{(m)} = \overline{X} \times \{b\}$. In particular, for every component Z' of $L^+(\overline{Z})$, the scheme $\pi_j(Z')$ is an fs-cycle which is smooth along $X \times \widehat{B}_j$ for $1 \leq j \leq m$.

To show smoothness of $\pi_j(Z')$ along $\{x\} \times \widehat{B}_j$ for $m+1 \leq j \leq n$, it is equivalent to show using Claim 1, that the components of $L^+(\overline{Z}^{(j)})$ are smooth along $\{x\} \times \widehat{B}_j$ for $m+1 \leq j \leq n$. We do it after Claim 2 below.

Set $D_x = \phi^{-1}(y)$, fix a point $\alpha = (a, b) \in D_x \times \widehat{B}_j$ and set $\beta = \widehat{\phi}_j(\alpha) = (y, b)$. We write $D_x^b = D_x \times \{b\} = \widehat{\phi}_j^{-1}(\beta)$. Set $W_1^{(j)} = U \times \widehat{B}_j$ and $W_2^{(j)} = \phi^{-1}(U) \times \widehat{B}_j$. Notice that $W_2^{(j)}$ is smooth by the property (2).

Claim 2: *If $\alpha \in \overline{Z}^{(j)}$, then $\overline{Z}^{(j)}$ is the only irreducible component of $\widetilde{Z}^{(j)}$ passing through α .*

(\because) Since $\widehat{\phi}_j$ is finite and étale over $U \times \widehat{B}_j$, we see that the map $\mathcal{O}_{W_1^{(j)}, \beta} \rightarrow \mathcal{O}_{W_2^{(j)}, D_x^b}$ is finite and étale. In particular, the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\widetilde{Z}^{(j)}, D_x^b}$ is finite and étale. This in turn implies that the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\overline{Z}^{(j)}, D_x^b}$ is finite and unramified.

Since $j > m$, we see that $\overline{Z}^{(j)} \not\subset \overline{X} \times \{b'\}$ for any $b' \in \widehat{B}_j$. It follows from the second part of the condition (4) in Definition 10.1.1 and the property (6) in Proposition 10.3.2 that the map $\mathcal{O}_{\overline{Z}^{(j)}, D_x^b} \rightarrow \mathcal{O}_{\overline{Z}^{(j)}, \alpha}$ is an isomorphism. Combining this with the property (4) of Proposition 10.3.2, we conclude that the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\overline{Z}^{(j)}, \alpha}$ is an injective (since $\overline{Z}^{(j)} \twoheadrightarrow T^{(j)}$), finite and unramified map of local rings of closed points of affine integral domains over k , which induces an isomorphism of the residue fields. It follows from Lemma 7.1.2 that the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\overline{Z}^{(j)}, \alpha}$ is an isomorphism.

On the other hand, the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\widetilde{Z}^{(j)}, D_x^b}$, being finite and étale, shows that the map $\mathcal{O}_{T^{(j)}, \beta} \rightarrow \mathcal{O}_{\widetilde{Z}^{(j)}, \alpha}$ is étale. In particular, the map $\widehat{\mathcal{O}}_{T^{(j)}, \beta} \rightarrow \widehat{\mathcal{O}}_{\widetilde{Z}^{(j)}, \alpha}$ of completions is finite and étale. Since it induces an isomorphism on the level of the residue fields, it follows again from Lemma 7.1.2 that the map $\widehat{\mathcal{O}}_{T^{(j)}, \beta} \rightarrow \widehat{\mathcal{O}}_{\overline{Z}^{(j)}, \alpha}$ is an isomorphism.

Hence, there are local homomorphisms of complete local rings

$$(11.2) \quad \widehat{\mathcal{O}}_{T^{(j)},\beta} \rightarrow \widehat{\mathcal{O}}_{\widetilde{Z}^{(j)},\alpha} \twoheadrightarrow \widehat{\mathcal{O}}_{\overline{Z}^{(j)},\alpha},$$

where both the first map and the composite map are isomorphisms. Thus, the second map is an isomorphism too. The second map in (11.2) being a priori a surjection, the Krull intersection theorem ([36, Theorem 8.10]) says that the second map is an isomorphism if and only if the map $\mathcal{O}_{\widetilde{Z}^{(j)},\alpha} \twoheadrightarrow \mathcal{O}_{\overline{Z}^{(j)},\alpha}$ (without completion) is an isomorphism. This in turn is equivalent to that $\overline{Z}^{(j)}$ is the only irreducible component of $\widetilde{Z}^{(j)}$ passing through α . The Claim 2 is now proven.

Going back, we fix a component Z' of $L^+(\overline{Z}^{(j)})$, and we prove that Z' is smooth along $\{x\} \times \widehat{B}_j$.

Let $\alpha' = (x, b) \in Z'$ for some $b \in \widehat{B}_j$. Set $\beta = (y, b) \in U \times \widehat{B}_j$. We have shown in the course of Claim 2 that the map $\widehat{\mathcal{O}}_{T^{(j)},(y,b)} \rightarrow \widehat{\mathcal{O}}_{\widetilde{Z}^{(j)},(a,b)}$ of completions in (11.2) is an isomorphism for every $a \in D_x$. Since $\alpha' \in Z' \subset \widetilde{Z}^{(j)}$, there must exist a point $\alpha = (a, b) \in \overline{Z}^{(j)}$ for some $a \in D_x$. Since Z' is a residual component, Claim 2 implies that we must have $a \neq x$. In this case, Claim 2 again shows that $\overline{Z}^{(j)}$ is the only irreducible component of $\widetilde{Z}^{(j)}$ passing through α . In particular, $\mathcal{O}_{\widetilde{Z}^{(j)},\alpha} \xrightarrow{\sim} \mathcal{O}_{\overline{Z}^{(j)},\alpha}$.

Since $a \neq x$, it follows from the condition (3) in Definition 10.1.1 and the property (6) in Proposition 10.3.2 that $\mathcal{O}_{\overline{Z}^{(j)},\alpha}$ is smooth, thus so is $\mathcal{O}_{\widetilde{Z}^{(j)},\alpha}$. Using the isomorphisms $\widehat{\mathcal{O}}_{T^{(j)},\beta} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widetilde{Z}^{(j)},\alpha}$ and $\widehat{\mathcal{O}}_{T^{(j)},\beta} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widetilde{Z}^{(j)},\alpha'}$, we conclude that $\mathcal{O}_{T^{(j)},\beta}$ and $\mathcal{O}_{\widetilde{Z}^{(j)},\alpha'}$ are both smooth. Since $\alpha' \in Z'$, the local ring $\mathcal{O}_{\widetilde{Z}^{(j)},\alpha'}$ will be smooth only if Z' is the only component of $\widetilde{Z}^{(j)}$ passing through $\alpha' = (x, b)$ and is smooth at this point. The proof of the Proposition 11.1.1 is now complete. \square

Remark 11.1.2. We have in fact shown that if a point $\alpha = (x, b)$ lies in any irreducible component Z' of $L^+(\overline{Z})$, then Z' is the only component passing through α . Moreover, Z' is smooth at α if $Z' \neq \overline{Z}_i$ for all $1 \leq i \leq s$. The same holds when Z is replaced by $Z^{(j)}$.

Remark 11.1.3. Observe that Claim 2 in the proof of Proposition 11.1.1 can be deduced from Lemma 7.1.3. We gave another argument because from there we can deduce a stronger assertion than that of Lemma 7.1.3 regarding smoothness of the residual components.

Corollary 11.1.4. *Let Z be an fs-cycle on $X \times \widehat{B}_n$ with the irreducible components $\{Z_1, \dots, Z_s\}$. Suppose each Z_i is finite surjective over X , but Z is not necessarily rectifiable. Then there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$ such that the following hold:*

(1) *The linear projection away from L defines a Cartesian square*

$$(11.3) \quad \begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{A}_k^r & \hookrightarrow & \mathbb{P}_k^r \end{array}$$

such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subset \mathbb{A}_k^r$.

- (2) The map $\phi : X \rightarrow \mathbb{A}_k^r$ is étale over an affine neighborhood U of $\phi(\Sigma)$.
- (3) For each $1 \leq i \leq s$, one of the following holds:
 - (A) Z_i is smooth and $Z_i = \phi^{-1}(\phi(Z_i))_{\text{red}}$.
 - (B) For each irreducible component Z' of $L^+(Z_i)$, and each $1 \leq j \leq n$, the scheme $\pi_j(Z')$ is an fs-cycle which is also smooth at all points over $\Sigma \times \widehat{B}_j$.

Proof. We write $Z = W_1 + W_2$, where W_1 is rectifiable and W_2 is not rectifiable. We now apply Proposition 11.1.1 to get a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$ such that (1), (2) and (3)(B) are satisfied for W_1 .

Let Z_i be a component of W_2 , so it is not rectifiable. This means that $Z_i \subset X \times \{b_i\}$ for some closed point $b_i \in \widehat{B}_n$. Since Z_i is an irreducible scheme of dimension r which is finite and surjective over X , we must have $Z_i = X \times \{b_i\}$. In particular, Z_i is smooth and $\overline{Z}_i = \overline{X} \times \{b_i\}$. But this means that $\phi(\overline{Z}_i) = \mathbb{P}_k^r \times \{b_i\}$ and hence $\phi^{-1}(\phi(\overline{Z}_i))_{\text{red}} = \overline{X} \times \{b_i\} = \overline{Z}_i$. This proves the corollary. \square

11.2. The proof.

Theorem 11.2.1. *Let $V = \text{Spec}(R)$ be an r -dimensional smooth semi-local k -scheme of geometric type with the set Σ of closed points. Suppose that $r, m, n \geq 1$. Let $\alpha \in \text{Tz}_{\text{fs}}^n(V, n; m)$ be a cycle. Then we can find:*

- (1) an affine atlas (X, Σ) for V and a finite surjective map $\phi : X \rightarrow \mathbb{A}_k^r$,
- (2) a cycle $\overline{\alpha} \in \text{Tz}_{\Sigma}^n(X, n; m)$ with $\alpha = \overline{\alpha}_V$, and
- (3) an open subset $U \subset \mathbb{A}_k^r$ containing $\phi(\Sigma)$ with a finite and étale map $\phi : \phi^{-1}(U) \rightarrow U$

such that for every component Z of $\overline{\alpha}$, one of the following holds:

- (A) Z is smooth and $Z = \phi^{-1}(\phi(Z))_{\text{red}}$.
- (B) $\phi^+([Z])$ is an sfs-cycle over V .

Proof. By Lemmas 2.4.1 and 2.4.3, we can find an affine atlas (X, Σ) for V and a cycle $\overline{\alpha} \in \text{Tz}_{\Sigma}^n(X, n; m)$ satisfying (1) and (2). By Lemma 5.1.5 and Proposition 5.1.6, after shrinking X , we may assume that each irreducible component of $\overline{\alpha}$ is finite and surjective over X . Choose a closed embedding $X \hookrightarrow \mathbb{A}_k^m$ and the closure $\overline{X} \hookrightarrow \mathbb{P}_k^m$ as before.

We choose a closed embedding $\overline{X} \hookrightarrow \mathbb{P}_k^N$ with $N \gg r$ such that for a general linear subspace $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N)$, the assertion of Corollary 11.1.4 holds.

Let $\phi : \phi^{-1}(U) \rightarrow U$ be the map of smooth affine schemes as in Corollary 11.1.4. Let $V' = \text{Spec}(\mathcal{O}_{U, \phi(\Sigma)}) = \text{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \phi(\Sigma)})$. Since $\phi^{-1}(\phi(\Sigma)) \subset X$, we see that the restriction of ϕ on V' is finite and étale. It follows that ϕ is finite (and hence flat) over an affine neighborhood of $\phi(\Sigma)$ in U . It follows now from [16, Exercise III-10.2, p.275] that ϕ is finite and étale over an affine neighborhood of $\phi(\Sigma)$ in U . So, we get an affine atlas (X, Σ) for V satisfying (1) and (2). The last assertion follows directly by applying the property (3) in Corollary 11.1.4 to $\overline{\alpha}$. \square

Finally, we get to the conclusion of §5 ~ 11:

Proof of Theorem 5.2.3. From the definition of $\mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m)$, the map sfs_V is injective. We prove its surjectivity. By Theorem 7.4.2, we may replace $\mathrm{TCH}^n(V, n; m)$ by $\mathrm{TCH}_{\mathrm{sfs}}^n(V, n; m)$.

Let $\alpha \in \mathrm{Tz}_{\mathrm{sfs}}^n(V, n; m)$ be an fs-cycle with $\partial(\alpha) = 0$. Apply Lemma 2.4.3 to choose an affine atlas (X, Σ) for V and a cycle $\bar{\alpha} \in \mathrm{Tz}^n(X, n; m)$ such that $\partial(\alpha) = 0$. If $X \simeq \mathbb{A}_k^r$, we can apply Theorem 6.2.1 to write $\alpha = \beta + \partial(\gamma)$, where $\beta \in \mathrm{Tz}_{\mathrm{sfs}}^n(V, n; m) \subset \mathrm{Tz}_{\mathrm{sfs}}^n(V, n; m)$ and $\gamma \in \mathrm{Tz}^n(V, n+1; m)$. This solves the problem in this case.

If X is not an affine space, we let $\phi : X \rightarrow \mathbb{A}_k^r$ be the finite and flat map as in Theorem 11.2.1 and let $\Sigma' = \phi(\Sigma)$. Let $V' = \mathrm{Spec}(\mathcal{O}_{\mathbb{A}_k^r, \Sigma'})$ and let $W = X \times_{\mathbb{A}^r} V'$. So, we have inclusions $\Sigma \subset V \hookrightarrow W \hookrightarrow X$, and a finite and flat morphism $\phi_\Sigma : W \rightarrow V'$ of regular semi-local schemes, where V' is $\phi_*(\bar{\alpha}_W)$ -linear. Let $j : V \rightarrow W$ be the localization map.

We can write $\bar{\alpha}_W = (\bar{\alpha}_W - \phi^*\phi_*(\bar{\alpha}_W)) + \phi^*\phi_*(\bar{\alpha}_W)$. We also have $\partial(\phi_*(\bar{\alpha}_W)) = \phi_*(\partial(\bar{\alpha}_W)) = 0$. Since V' is $\phi_*(\bar{\alpha}_W)$ -linear, by the previous case we can write $\phi_*(\bar{\alpha}_W) = \eta_1 + \partial(\eta_2)$, where $\eta_1 \in \mathrm{Tz}_{\mathrm{sfs}}^n(V', n; m)$ and $\eta_2 \in \mathrm{Tz}^n(V', n+1; m)$. This yields $\phi^*\phi_*(\bar{\alpha}_W) = \phi^*(\eta_1) + \partial(\phi^*(\eta_2))$. Since $\phi : W \rightarrow V'$ is finite and étale, ϕ^* preserves the sfs-cycles. In particular, $\phi^*(\eta_1) \in \mathrm{Tz}_{\mathrm{sfs}}^n(W, n; m)$.

It follows from Theorem 11.2.1 that $j^*(\bar{\alpha}_W - \phi^*\phi_*(\bar{\alpha}_W)) \in \mathrm{Tz}_{\mathrm{sfs}}^n(V, n; m)$. Let $\beta = j^*(\bar{\alpha}_W - \phi^*\phi_*(\bar{\alpha}_W)) + j^*(\phi^*(\eta_1))$ and $\gamma = j^*(\phi^*(\eta_2))$. Then, we get

$$\begin{aligned} \alpha &= j^*(\bar{\alpha}_W) = j^*(\bar{\alpha}_W - \phi^*\phi_*(\bar{\alpha}_W)) + j^*\phi^*(\eta_1) + j^*(\partial(\phi^*(\eta_2))) \\ &= j^*(\bar{\alpha}_W - \phi^*\phi_*(\bar{\alpha}_W)) + j^*\phi^*(\eta_1) + \partial(j^*\phi^*(\eta_2)) = \beta + \partial(\gamma), \end{aligned}$$

with $\beta \in \mathrm{Tz}_{\mathrm{sfs}}^n(V, n; m)$ and $\gamma \in \mathrm{Tz}^n(V, n+1; m)$. Since $\partial(\alpha) = 0$, we must have $\partial(\beta) = 0$ as well. This proves the theorem. \square

12. SURJECTIVITY OF THE DE RHAM-WITT-CHOW HOMOMORPHISM

Let k be any perfect field of characteristic $\neq 2$, unless stated otherwise. The goal of this section is to prove the surjectivity of the de Rham-Witt-Chow homomorphism (4.3) for regular semi-local k -schemes of geometric type to complete the proof of Theorem 4.2.2. In view of Theorem 5.2.3, we need to show that every sfs-cycle is generated by cycles that are *Witt-Milnor cycles over R* , or *symbolic over R* , that is, in the image of the map $\tau_{n,m}^R$. This will be achieved by a delicate usage of the Witt-complex structure on the additive higher Chow groups of regular semi-local k -schemes.

12.1. Traceability of de Rham-Witt forms via cycles.

12.1.1. Notion of traceability. Let $R \rightarrow S$ be a finite extension of regular semi-local k -algebras essentially of finite type and let $f : \mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$ be the corresponding morphism of schemes. Let $m, n \geq 1$ be two integers. Given this, we obtain a diagram:

$$(12.1) \quad \begin{array}{ccc} \mathbb{W}_m \Omega_S^{n-1} & \xrightarrow{\tau_{n,m}^S} & \mathrm{TCH}^n(S, n; m) \\ & & \downarrow f_* \\ \mathbb{W}_m \Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m). \end{array}$$

A problem one faces is that it is *a priori* unknown if there is a trace map $\mathbb{W}_m \Omega_S^{n-1} \rightarrow \mathbb{W}_m \Omega_R^{n-1}$ that completes (12.1). The main idea here is to use the push-forward f_* as a *trace-like* operation on the de Rham-Witt forms via algebraic cycles.

Definition 12.1.1. Let $R \rightarrow S$ be as above. We say that a de Rham-Witt form $\omega \in \mathbb{W}_m \Omega_S^{n-1}$ is *traceable to R* (via cycles) if $f_* \circ \tau_{n,m}^S(\omega) \in \text{Image}(\tau_{n,m}^R)$.

12.1.2. *Traceability under simple ring extensions.* We aim to show in Proposition 12.1.5 the traceability of the de Rham-Witt forms under “simple extensions” for regular semi-local rings. Recall:

Definition 12.1.2. A ring extension $R \subset S$ is said to be a *simple extension* if there exists a monic irreducible polynomial $p(t) \in R[t]$ such that $S \simeq R[t]/(p(t))$.

Let $e = \deg(p(t))$. Let $a := t \bmod (p(t))$ in S . Then S is a finite free extension of R with an R -basis $\{1, a, a^2, \dots, a^{e-1}\}$. We need the following basic fact about the ring of Witt vectors.

Lemma 12.1.3. *Let S be a free R -algebra with $\{x_1, \dots, x_e\}$ as an R -basis. Let T be a finite truncation set. Then every $\omega \in \mathbb{W}_T(S)$ is uniquely written as*

$$\omega = \sum_{n \in T} \sum_{i=1}^e V_n([c_{n,i}]_{T/n} \cdot [x_i]_{T/n}),$$

for some $c_{n,i} \in R$, where $[-]_{T/n}$ denotes the Teichmüller lift in $\mathbb{W}_{T/n}(R)$, and V_n is the n -th Verschiebung operator.

Proof. Its proof is similar to that of [41, Lemma 2.20], for instance. Let $\omega = (\omega_n)_{n \in T} \in \mathbb{W}_T(S)$. Suppose $\omega \neq 0$, for otherwise there is nothing to prove. Define an operator φ as follows: first choose $s_0 = \min\{s \in T \mid \omega_s \neq 0\}$. This minimum exists because $\omega \neq 0$. Here, $\omega_{s_0} \in S$ so that there exists a unique expression $\omega_{s_0} = \sum_{i=1}^e c_{s_0,i} \cdot x_i$ in S for some $c_{s_0,i} \in R$. Now, define $\varphi(\omega) := \omega - V_{s_0}(\sum_{i=1}^e [c_{s_0,i}]_{T/s_0} \cdot [x_i]_{T/s_0})$. Now, we have either $\varphi(\omega) = 0$, or $\varphi(\omega) \neq 0$. In the former case, the argument stops, while in the latter case, there exists $s_1 := \min\{s \in T \mid \varphi(\omega)_s \neq 0\}$ and by construction $s_1 > s_0$. We repeat this process. Since $|T| < \infty$, there exists $N \geq 1$ such that eventually $\varphi^N(\omega) = 0$. \square

When S is a simple extension, and $T = \{1, \dots, m\}$, we immediately deduce $\omega = \sum_{i=1}^m \sum_{j=0}^{e-1} V_i([c_{i,j}]_{[m/i]} \cdot [a]_{[m/i]}^j)$.

Recall from [41, Proposition A.9] that for a finite free extension of rings $R \rightarrow S$, and $m \geq 1$, there is a trace map $\text{Tr}_{S/R} : \mathbb{W}_m(S) \rightarrow \mathbb{W}_m(R)$ which commutes with the Frobenius and the Verschiebung operators, and satisfies other usual properties of the trace maps. This $\text{Tr}_{S/R}$ is given as follows: for the finite free extension $R[[t]] \rightarrow S[[t]]$, we have the norm map $N_{S/R} : (S[[t]])^\times \rightarrow (R[[t]])^\times$ given by the determinant of the left multiplication maps, which induces $N_{S/R} : (1+tS[[t]])^\times / (1+t^{m+1}S[[t]])^\times \rightarrow (1+tR[[t]])^\times / (1+t^{m+1}R[[t]])^\times$. This $N_{S/R}$ is the definition of $\text{Tr}_{S/R}$ via the identification (3.1).

Lemma 12.1.4. *Let $R \subset S$ be a simple extension of regular semi-local k -algebras essentially of finite type and let $m \geq 1$ be an integer. Then the diagram*

$$(12.2) \quad \begin{array}{ccc} \mathbb{W}_m(S) & \xrightarrow{\tau_{1,m}^S} & \mathrm{TCH}^1(S, 1; m) \\ \mathrm{Tr}_{S/R} \downarrow & & \downarrow f_* \\ \mathbb{W}_m(R) & \xrightarrow{\tau_{1,m}^R} & \mathrm{TCH}^1(R, 1; m) \end{array}$$

commutes, where $f : \mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$ is the induced map.

Proof. Using Proposition 4.1.1 and Lemma 12.1.3, it only remains to check that $\tau_{1,m}^R(\mathrm{Tr}_{S/R}([x])) = f_*([\Gamma_{(1-xt)}])$ for all $x \in S$, where $\Gamma_{(1-xt)}$ is the cycle in $\mathrm{TCH}^1(S, 1; m)$ corresponding to the ideal $(1 - xt) \subset S[t]$. Since $[x] \in \mathbb{W}_m(S)$ corresponds to $1 - xt \in (1 + tS[[t]])^\times / (1 + t^{m+1}S[[t]])^\times$, by the definition of $\mathrm{Tr}_{S/R}$ we have $\mathrm{Tr}_{S/R}([x]) = N_{S/R}(1 - xt)$. On the other hand, for a polynomial representative $g(t) \in (1 + tR[[t]])^\times / (1 + t^{m+1}R[[t]])^\times$, we have by definition $\tau_{1,m}^R(g(t)) = [\Gamma_{(g(t))}]$. Hence, $\tau_{1,m}^R(\mathrm{Tr}_{S/R}([x])) = \tau_{1,m}^R(N_{S/R}(1 - xt)) = [\Gamma_{(N_{S/R}(1 - xt))}]$. On the other hand, by [11, Proposition 1.4(2)], we have $f_*(\mathrm{div}(1 - xt)) = [\mathrm{div}(N_{S/R}(1 - xt))]$. Since $1 - xt$ and $N_{S/R}(1 - xt)$ are regular functions, we have $\mathrm{div}(1 - xt) = \Gamma_{(1-xt)}$ and $\mathrm{div}(N_{S/R}(1 - xt)) = \Gamma_{(N_{S/R}(1 - xt))}$. Hence, we have $[\Gamma_{(N_{S/R}(1 - xt))}] = f_*([\Gamma_{(1-xt)}])$. Hence, we have $\tau_{1,m}^R(\mathrm{Tr}_{S/R}([x])) = [\Gamma_{(N_{S/R}(1 - xt))}] = f_*([\Gamma_{(1-xt)}])$, as desired. \square

Proposition 12.1.5. *Let $R \rightarrow S$ be a simple extension of regular semi-local k -algebras essentially of finite type and let $m, n \geq 1$ be integers. Then all elements in $\mathbb{W}_m \Omega_S^{n-1}$ are traceable to R .*

Proof. Let $p(t) \in R[t]$ be a monic polynomial of degree e such that $S \simeq R[t]/(p(t))$. Let $a = t \bmod p(t)$ so that $\{1, a, \dots, a^{e-1}\}$ is an R -basis of S . For $m, n \geq 1$, let $P_{n,m}$ be the statement

$P_{n,m}$: all elements in $\mathbb{W}_m \Omega_S^{n-1}$ are traceable to R .

We prove the proposition by a double induction argument on the variables $(n, m) \in \mathbb{N} \times \mathbb{N}$. We begin with the boundary cases:

Case 1: We show first that $P_{1,m}$ and $P_{n,1}$ are true.

Note that the statement $P_{1,m}$ holds by Lemma 12.1.4. In particular, $P_{1,1}$ is also true.

Subcase 1-1: To show $P_{2,1}$, note that every element of $\mathbb{W}_1 \Omega_S^1 \simeq \Omega_{S/\mathbb{Z}}^1$ is a finite sum of 1-forms of the type $ca^i d(c'^j) = ca^{i+j} dc' + jcc' a^{i+j-1} da$ for some $c, c' \in R$. So, we reduce to show that 1-forms of the types $ca^i dc'$ and $ca^i da$ are traceable for all $c, c' \in R$ and $i \geq 0$.

For $ca^i dc'$, we have

$$(12.3) \quad \begin{aligned} f_* \circ \tau_{2,1}^S(ca^i dc') &= {}^\dagger f_* (\tau_{1,1}^S(a^i) \cdot \tau_{2,1}^S(cdc')) = f_* (\tau_{1,1}^S(a^i) \cdot \tau_{2,1}^S(f^*(cdc'))) \\ &= {}^\dagger f_* (\tau_{1,1}^S(a^i) \cdot f^*(\tau_{2,1}^R(cdc'))) = {}^1 f_* (\tau_{1,1}^S(a^i)) \cdot \tau_{2,1}^R(cdc') \\ &= {}^2 \tau_{1,1}^R(\mathrm{Tr}_{S/R}(a^i)) \cdot \tau_{2,1}^R(cdc') = {}^\dagger \tau_{2,1}^R(\mathrm{Tr}_{S/R}(a^i) \cdot (cdc')). \end{aligned}$$

Here, the equalities $= {}^\dagger$ hold because $\tau_{m,n}^R, \tau_{m,n}^S$ are morphisms of DGAs. The equality $= {}^\dagger$ holds by Corollary 4.1.4, $= {}^1$ by the projection formula for the additive

higher Chow groups and $=^2$ holds by Lemma 12.1.4. We conclude that 1-forms of the type $ca^i dc'$ are traceable to R .

Next, for $ca^i da$, note that by the part of definition of a restricted Witt-complex in §2.5.2(v), we can write $ca^i da = cF_{i+1}d[a]$, using the Frobenius operator. Hence,

$$\begin{aligned}
 (12.4) \quad & f_* \circ \tau_{2,1}^S(ca^i da) = f_* \circ \tau_{2,1}^S(cF_{i+1}d[a]) = f_* \circ \tau_{2,1}^S(f^*(c)F_{i+1}d[a]) \\
 & =^0 f_*(\tau_{1,1}^S(f^*(c)) \cdot \tau_{2,1}^S(F_{i+1}d[a])) =^\dagger f_*(\tau_{1,1}^S(f^*(c)) \cdot F_{i+1}\delta\tau_{1,2i+1}^S([a])) \\
 & =^\dagger f_* (f^*(\tau_{1,1}^R(c)) \cdot F_{i+1}\delta\tau_{1,2i+1}^S([a])) =^1 \tau_{1,1}^R(c) \cdot f_* F_{i+1}\delta\tau_{1,2i+1}^S([a]) \\
 & =^2 \tau_{1,1}^R(c) \cdot F_{i+1}\delta f_* \tau_{1,2i+1}^S([a]) =^3 \tau_{1,1}^R(c) \cdot F_{i+1}\delta\tau_{1,2i+1}^R(\text{Tr}_{S/R}([a])) \\
 & =^\dagger \tau_{1,1}^R(c) \cdot F_{i+1}\tau_{2,2i+1}^R(d(\text{Tr}_{S/R}([a]))) =^\dagger \tau_{1,1}^R(c) \cdot \tau_{2,1}^R F_{i+1}d(\text{Tr}_{S/R}([a])) \\
 & =^0 \tau_{2,1}^R(c \cdot F_{i+1}d(\text{Tr}_{S/R}(a))),
 \end{aligned}$$

where the equalities $=^\dagger$ hold by (4.4), the equality $=^\dagger$ holds by Corollary 4.1.4, $=^1$ holds by the projection formula for f_* and f^* , $=^2$ holds by Proposition 4.1.1 and $=^3$ holds by Lemma 12.1.4, and the equalities $=^0$ holds because $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are morphisms of DGAs. We conclude that 1-forms of the type $ca^i da$ are traceable to R . Hence $P_{2,1}$ is true.

Subcase 1-2: Suppose now that $n > 2$ and that the statements $P_{i,1}$ are true for all $1 \leq i < n$. It suffices to show that the forms of the type $\omega = c_0 a^{i_0} d(c_1 a^{i_1}) \wedge \cdots \wedge d(c_{n-1} a^{i_{n-1}})$ is traceable to R , where $c_0, \dots, c_{n-1} \in R$ and $i_0, \dots, i_{n-1} \geq 0$ are integers. Each $d(c_j a^{i_j})$ is equal to $a^{i_j} dc_j + i_j c_j a^{i_j-1} da$ by the Leibniz rule, so that expanding the terms of ω , we reduce to show that every element of the form

$$\omega_0 := c_0 a^i dc_1 \wedge \cdots \wedge dc_s \wedge \underbrace{da \wedge \cdots \wedge da}_{n-s-1}$$

is traceable to R , where $0 \leq s \leq n-1$ and $c_0, \dots, c_s \in R$.

- If $n-s-1 = 0$, then the traceability of ω follows by repeating the steps in (12.3) in verbatim.

- If $n-s-1 = 1$, let $\omega'_0 := a^i da$ so that $\omega_0 = c_0 dc_1 \wedge \cdots \wedge dc_s \wedge \omega'_0$. Since $P_{2,1}$ is true, we can write $f_* \tau_{2,1}^S(\omega'_0) =^\spadesuit \tau_{2,1}^R(\omega''_0)$ for some $\omega''_0 \in \Omega_{R/\mathbb{Z}}^1$. Set $\eta := c_0 dc_1 \wedge \cdots \wedge dc_s \in \Omega_{R/\mathbb{Z}}^s$. Then, we have

$$\begin{aligned}
 (12.5) \quad & f_* \circ \tau_{n,1}^S(\omega_0) = f_* \circ \tau_{n,1}^S(\eta \wedge \omega'_0) = f_* \circ \tau_{n,1}^S(f^*(\eta) \wedge \omega'_0) \\
 & =^\dagger f_*(\tau_{n-1,1}^S(f^*(\eta)) \wedge \tau_{2,1}^S(\omega'_0)) =^\dagger f_*(f^* \tau_{n-1,1}^R(\eta) \wedge \tau_{2,1}^S(\omega'_0)) \\
 & =^1 \tau_{n-1,1}^R(\eta) \wedge f_* \tau_{2,1}^S(\omega'_0) =^\spadesuit \tau_{n-1,1}^R(\eta) \wedge \tau_{2,1}^R(\omega''_0) =^\dagger \tau_{n,1}^R(\eta \wedge \omega''_0),
 \end{aligned}$$

where the equalities $=^\dagger$ hold because $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are morphisms of DGAs, $=^\dagger$ holds by Corollary 4.1.4, and $=^1$ holds by the projection formula for f_* and f^* . We conclude that ω_0 is traceable to R .

- If $n-1-s > 1$, then $\omega_0 = 0$ since $2 \in S^\times$, so ω_0 is traceable to R .

We have thus shown that $P_{n,1}$ and $P_{1,m}$ are true for all $n, m \geq 1$.

Case 2: The general case. We now show that $P_{n,m}$ is true in general by using double induction on (n, m) . Fix $m, n \geq 2$ and suppose that we know $P_{i,j}$ holds for all $1 \leq i \leq n$, $1 \leq j \leq m$, except $(i, j) = (n, m)$.

Through the surjection $\Omega_{\mathbb{W}_m(S)}^{n-1} \rightarrow \mathbb{W}_m \Omega_S^{n-1}$ and Lemma 12.1.3, we know that every element in $\mathbb{W}_m \Omega_S^{n-1}$ is a sum of de Rham-Witt forms of the type $\omega = V_{r_0}([c_0][a]^{i_0}) \cdot dV_{r_1}([c_1][a]^{i_1}) \wedge \cdots \wedge dV_{r_{n-1}}([c_{n-1}][a]^{i_{n-1}})$, where $c_0, \dots, c_{n-1} \in R$, $r_0, \dots, r_{n-1} \in \{1, \dots, m\}$, and $0 \leq i_0, \dots, i_{n-1} \leq e-1$.

Subcase 2-1: First, consider the case $r_0 > 1$. Let $\omega_0 := dV_{r_1}([c_1][a]^{i_1}) \wedge \cdots \wedge dV_{r_{n-1}}([c_{n-1}][a]^{i_{n-1}})$. In this case, we can write

$$\omega = V_{r_0}([c_0][a]^{i_0}) \cdot \omega_0 = V_{r_0}([c_0][a]^{i_0} \cdot F_{r_0}(\omega_0))$$

by the projection formula for V_r and F_r (see §3.3(iii)). Since $\omega'_0 := [c_0][a]^{i_0} \cdot F_{r_0}(\omega_0) \in \mathbb{W}_{[m/r_0]}\Omega_S^{n-1}$, it is traceable to R by the induction hypothesis $P_{n,[m/r_0]}$. In particular, there exists $\eta \in \mathbb{W}_{[m/r_0]}\Omega_R^{n-1}$ such that $f_*\tau_{n,[m/r_0]}^S(\omega'_0) = \clubsuit \tau_{n,[m/r_0]}^R(\eta)$. This in turn yields

$$\begin{aligned} f_*\tau_{n,m}^S(\omega) &= f_*\tau_{n,m}^S V_{r_0}(\omega'_0) =^\dagger f_*V_{r_0}\tau_{n,[m/r_0]}^S(\omega'_0) \\ &=^\ddagger V_{r_0}f_*\tau_{n,[m/r_0]}^S(\omega'_0) = \clubsuit V_{r_0}\tau_{n,[m/r_0]}^R(\eta) =^\dagger \tau_{n,m}^R(V_{r_0}\eta), \end{aligned}$$

where the equalities \dagger hold by (4.5), \ddagger holds by Proposition 4.1.1. This shows that ω is traceable to R .

Subcase 2-2: Suppose now that $r_0 = 1$, but for some $j > 0$, we have $r_j > 1$. We may assume that $r_1 > 1$, without loss of generality. We let $\omega_0 := dV_{r_2}([c_2][a]^{i_2}) \wedge \cdots \wedge dV_{r_{n-1}}([c_{n-1}][a]^{i_{n-1}})$. By the Leibniz rule, we have

$$\begin{aligned} V_{r_0}([c_0][a]^{i_0}) \cdot dV_{r_1}([c_1][a]^{i_1}) &= [c_0][a]^{i_0} \cdot dV_{r_1}([c_1][a]^{i_1}) \\ &= d([c_0][a]^{i_0} \cdot V_{r_1}([c_1][a]^{i_1})) - V_{r_1}([c_1][a]^{i_1}) \cdot d([c_0][a]^{i_0}). \end{aligned}$$

Hence, $\omega = \omega_1 - \omega_2$, where $\omega_1 := d([c_0][a]^{i_0} \cdot V_{r_1}([c_1][a]^{i_1})) \wedge \omega_0$ and $\omega_2 = V_{r_1}([c_1][a]^{i_1}) \cdot d([c_0][a]^{i_0}) \wedge \omega_0$. Let $\omega'_1 := [c_0][a]^{i_0} \cdot V_{r_1}([c_1][a]^{i_1})$ so that $\omega_1 = d\omega'_1 \wedge \omega_0 = d(\omega'_1 \cdot \omega_0)$.

Since $\omega'_1 \cdot \omega_0 \in \mathbb{W}_m\Omega_S^{n-2}$, it follows by the induction hypothesis $P_{n-1,m}$ that there is an element $\eta \in \mathbb{W}_m\Omega_R^{n-2}$ such that $f_*\tau_{n-1,m}^S(\omega'_1 \cdot \omega_0) =^\heartsuit \tau_{n-1,m}^R(\eta)$. Thus,

$$\begin{aligned} f_*\tau_{n,m}^S(\omega_1) &= f_*\tau_{n,m}^S(d(\omega'_1 \cdot \omega_0)) =^\dagger f_*\delta(\tau_{n-1,m}^S(\omega'_1 \cdot \omega_0)) \\ &=^\ddagger \delta f_*\tau_{n-1,m}^S(\omega'_1 \cdot \omega_0) =^\heartsuit \delta\tau_{n-1,m}^R(\eta) =^\dagger \tau_{n,m}^R(d\eta), \end{aligned}$$

where the equalities $=^\dagger$ hold by (4.4) because $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are morphism of DGAs, $=^\ddagger$ holds by Proposition 4.1.1. Hence ω_1 is traceable to R . Since ω_2 is of the form considered in *Subcase 2-1*, it is also traceable to R . Thus, ω is traceable to R .

Subcase 2-3: Now, the remaining case is when all $r_0 = r_1 = \cdots = r_{n-1} = 1$, i.e., $\omega = [c_0][a]^{i_0}d([c_1][a]^{i_1}) \wedge \cdots \wedge d([c_{n-1}][a]^{i_{n-1}})$. Its proof is almost identical to that of *Subcase 1-2*, which we argue now. Each $d([c_j][a]^{i_j})$ is equal to $[a]^{i_j}d[c_j] + i_j[c_j][a]^{i_j-1}d[a]$ by the Leibniz rule, so that expanding the terms of ω , we are reduced to show that elements of the form

$$(12.6) \quad \omega_0 := [c_0][a]^i d[c_1] \wedge \cdots \wedge d[c_s] \wedge \underbrace{d[a] \wedge \cdots \wedge d[a]}_{n-s-1}$$

are traceable, where $0 \leq s \leq n-1$ and $c_0, \dots, c_s \in R$.

- If $n-s-1 = 0$, then we can use Corollary 4.1.4, Lemma 12.1.4 and repeat the steps of (12.3) verbatim to conclude that ω_0 is traceable to R .

- If $n-s-1 = 1$, let $\omega'_0 := [a]^i d[a]$ so that $\omega_0 = [c_0]d[c_1] \wedge \cdots \wedge d[c_{n-2}] \wedge \omega'_0$. By the part of definition of a restricted Witt-complex in §2.5.2(v), we can write $\omega'_0 = [a]^i d[a] = F_{i+1}d[a]$. Set $\eta = [c_0]d[c_1] \wedge \cdots \wedge d[c_{n-2}] \in \mathbb{W}_m\Omega_R^{n-2}$, so that

$\omega_0 = \eta \wedge F_{i+1}d[a]$. (Remember, here $n \geq 2$.) This yields

$$\begin{aligned}
f_*\tau_{n,m}^S(\omega_0) &= f_*\tau_{n,m}^S(\eta \wedge F_{i+1}d[a]) = f_*\tau_{n,m}^S(f^*(\eta) \wedge F_{i+1}d[a]) \\
&=^\dagger f_*(\tau_{n-2,m}^S(f^*(\eta)) \wedge \tau_{2,m}^S F_{i+1}d[a]) =^\dagger f_*(f^*\tau_{n-2,m}^R(\eta) \wedge \tau_{2,m}^S F_{i+1}d[a]) \\
&=^0 \tau_{n-2,m}^R(\eta) \wedge f_*(\tau_{2,m}^S F_{i+1}d[a]) =^1 \tau_{n-2,m}^R \wedge f_*(F_{i+1}\tau_{2,(i+1)m+i}^S d[a]) \\
&=^1 \tau_{n-2,m}^R(\eta) \wedge f_*(F_{i+1}\delta\tau_{1,(i+1)m+i}^S([a])) =^2 \tau_{n-2,m}^R(\eta) \wedge F_{i+1}\delta f_*\tau_{1,(i+1)m+i}^S([a]) \\
&=^3 \tau_{n-2,m}^R(\eta) \wedge F_{i+1}\delta\tau_{1,(i+1)m+i}^R(\text{Tr}_{S/R}([a])) \\
&=^1 \tau_{n-2,m}^R(\eta) \wedge F_{i+1}\tau_{2,(i+1)m+i}^R d(\text{Tr}_{S/R}([a])) =^1 \tau_{n-2,m}^R(\eta) \wedge \tau_{2,m}^R F_{i+1}d(\text{Tr}_{S/R}([a])) \\
&=^\dagger \tau_{n,m}^R(\eta \wedge F_{i+1}d(\text{Tr}_{S/R}([a]))),
\end{aligned}$$

where the equalities $=^\dagger$ hold because $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are morphisms of DGAs, the equality $=^\dagger$ holds by Corollary 4.1.4, the equality $=^0$ is the projection formula for f_* and f^* , the equalities $=^1$ hold by (4.4), the equality $=^2$ holds by Proposition 4.1.1 and $=^3$ follows from Lemma 12.1.4. This shows that ω_0 is traceable to R .

- If $n - s - 1 > 1$, we set $\omega'_0 = \underbrace{d[a] \wedge \cdots \wedge d[a]}_{n-s-1}$. Since $2 \in S^\times$ and since

the Teichmüller lift is multiplicative, we see that $2 \in (\mathbb{W}_m(S))^\times$. In particular, $d[a] \wedge d[a] = 0$ in $\Omega_{\mathbb{W}_m(S)/\mathbb{Z}}^2$ and hence it is zero in $\mathbb{W}_m\Omega_S^2$. In particular, $\omega'_0 = 0$ in $\mathbb{W}_m\Omega_S^{n-s-1}$ so that $\omega_0 = 0$, which is traceable to R . We have thus shown that $P_{n,m}$ holds. The proof of the proposition is now complete. \square

12.2. Symbolicity of sfs-cycles and Proof of Theorem 4.2.2. In §12.2, we complete the proof of Theorem 4.2.2 using Theorem 5.2.3 and Proposition 12.1.5 as two key ingredients. We begin the following description of the de Rham-Witt-Chow homomorphism $\tau_{n,m}^R$ on the symbolic de Rham-Witt forms.

Lemma 12.2.1. *Let R be a regular semi-local k -algebra of geometric type. Let $a \in R$ and $b_i \in R^\times$ for $1 \leq i \leq n-1$. Then, $\tau_{n,m}^R([a]d\log[b_1] \wedge \cdots \wedge d\log[b_{n-1}]) = Z_{a,\underline{b}}$, where $Z_{a,\underline{b}} = \text{Spec}\left(\frac{R[t,y_1,\dots,y_{n-1}]}{(1-at,y_1-b_1,\dots,y_{n-1}-b_{n-1})}\right)$.*

Proof. This is an easy consequence of the fact that $\tau_{\bullet,m}^R$ is a morphism of restricted Witt-complexes over R (see § 4.2) and follows exactly by the method used in the computations in [32, (7.5)]. Indeed, by recursively applying the fact that $\tau_{\bullet,m}^R$ is a morphism of DGAs, it suffices to show the lemma when $n = 2$.

In this case, we have $\tau_{2,m}^R([a]d\log[b]) =^\dagger \tau_{2,m}^R([ab^{-1}]d[b]) = \tau_{1,m}^R([ab^{-1}]) \wedge (\delta \circ \tau_{1,m}^R([b]))$, where $=^\dagger$ holds because Teichmüller lift map is multiplicative. Using the definition of the differential δ on additive higher Chow groups (see [32, §6.1]), we have

$$\begin{aligned}
\tau_{1,m}^R([ab^{-1}]) \wedge (\delta \circ \tau_{1,m}^R([b])) &= \left[\text{Spec} \left(\frac{R[t]}{(1-ab^{-1}t)} \right) \right] \wedge \left[\delta \left(\text{Spec} \left(\frac{R[t]}{(1-bt)} \right) \right) \right] \\
&= \left[\text{Spec} \left(\frac{R[t]}{(1-ab^{-1}t)} \right) \right] \wedge \left[\text{Spec} \left(\frac{R[t,y]}{(1-bt,y-b)} \right) \right] \\
&= \Delta_R^* \left(\frac{(R \otimes_k R)[t,y]}{(1-(ab^{-1} \otimes b)t, y-(1 \otimes b))} \right)
\end{aligned}$$

and the last term is equal to $Z_{a,b}$ because $\Delta_R : \text{Spec}(R) \rightarrow \text{Spec}(R) \times \text{Spec}(R) \simeq \text{Spec}(R \otimes_k R)$ is induced by the product map $R \otimes_k R \rightarrow R$. \square

Proposition 12.2.2. *Let R be a regular semi-local k -algebra of geometric type, and let $m, n \geq 1$. Then every cycle in $\text{TCH}_{\text{sfs}}^n(R, n; m)$ is symbolic over R , that is, it is in the image of $\tau_{n,m}^R$.*

Proof. Let $[Z] \in \mathrm{Tz}_{\mathrm{sfs}}^n(R, n; m)$ be an irreducible sfs-cycle. By Proposition 5.1.8, we know that Z is a closed subscheme of $\mathrm{Spec}(R) \times \mathbb{A}_k^1 \times \overline{\square}_k^{n-1}$, which is in fact contained in $\mathrm{Spec}(R) \times \mathbb{A}_k^1 \times \mathbb{A}_k^{n-1}$. Moreover, $\partial Z = 0$ and the ideal $I(Z)$ of Z inside $R[t, y_1, \dots, y_{n-1}]$ is given by equations of the form:

$$(12.7) \quad \begin{cases} P(t) = 0, \\ Q_1(t, y_1) = 0, \\ \vdots \\ Q_{n-1}(t, y_1, \dots, y_{n-1}) = 0, \end{cases}$$

with the following additional properties: let $R_0 = R, R_1 = R[t]/(P(t))$ and $R_i = R_{i-1}[t_i]/(Q_{i-1})$ for $2 \leq i \leq n$. Then the rings $\{R_i\}_{1 \leq i \leq n}$ are all regular semi-local k -algebras such that each extension $R_{i-1} \subset R_i$ is simple. Let $f_i : \mathrm{Spec}(R_i) \rightarrow \mathrm{Spec}(R_{i-1})$ be the induced finite surjective map of semi-local schemes for $1 \leq i \leq n$. They are all flat by [16, Exercise III-10.9, p.276]. Let $f = f_1 \circ \dots \circ f_n$.

Let $c^{-1} := t \bmod I(Z)$ and $b_i := y_i \bmod I(Z)$ for $1 \leq i \leq n-1$. Note that a consequence of the sfs-property of Z is that $c^{-1}, b_i \in R_n^\times$ for all $1 \leq i \leq n-1$. Let $Z_n = \mathrm{Spec}\left(\frac{R_n[t, y_1, \dots, y_{n-1}]}{(1-ct, y_1-b_1, \dots, y_{n-1}-b_{n-1})}\right)$ and let $\eta_n := [c]d\log[b_1] \wedge \dots \wedge d\log[b_{n-1}]$. It follows then that $[Z_n] \in \mathrm{Tz}^n(R_n, n; m)$ such that $[Z] = f_*([Z_n])$.

It follows from Lemma 12.2.1 that $[Z_n] = \tau_{n,m}^{R_n}(\eta_n)$. Since f_n is a simple extension, by Proposition 12.1.5, we have $f_{n*}\tau_{n,m}^{R_n}(\eta_n) = \tau_{n,m}^{R_{n-1}}(\eta_{n-1})$ for some $\eta_{n-1} \in \mathbb{W}_m\Omega_{R_{n-1}}^{n-1}$. Since $f = f_1 \circ \dots \circ f_n$ and each f_i is a simple extension, by successive applications of f_{i*} and Proposition 12.1.5, we obtain $[Z] = f_*[Z_n] = \tau_{n,m}^R(\eta_0)$ for some $\eta_0 \in \mathbb{W}_m\Omega_R^{n-1}$. This means that $[Z]$ is symbolic over R , as desired. \square

Thus, we finally get to:

Proof of Theorem 4.2.2. If k is perfect, we have already shown in Corollary 4.3.2 that $\tau_{n,m}^R$ is injective. To show $\tau_{n,m}^R$ is surjective, first by Lemma 4.4.1, we reduce to the case when R is of geometric type. For this case, its surjectivity follows from Theorem 5.2.3 and Proposition 12.2.2. Suppose now that k is an imperfect field such that $\mathrm{char}(k) = p > 2$. We have shown in Proposition 4.2.1 that there exists a direct system of smooth semi-local \mathbb{F}_p -algebras R_i essentially of finite type with faithfully flat maps $\lambda_i : R_i \rightarrow R_{i+1}$ such that $\varinjlim_i R_i \xrightarrow{\sim} R$. Hence, we get a commutative diagram of restricted Witt-complexes over R :

$$(12.8) \quad \begin{array}{ccc} \varinjlim_i \mathbb{W}_m\Omega_{R_i}^{n-1} & \xrightarrow{\varinjlim_i \tau_{n,m}^{R_i}} & \varinjlim_i \mathrm{TCH}^n(R_i, n; m) \\ \downarrow & & \downarrow \\ \mathbb{W}_m\Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m). \end{array}$$

From the case of perfect base fields, the top horizontal arrow is an isomorphism. It follows from Lemma 2.4.6 that the right vertical arrow is an isomorphism. It follows from [41, Proposition 1.16] that the left vertical arrow is an isomorphism. We conclude that the bottom horizontal arrow is also an isomorphism. \square

Corollary 12.2.3. *Let k be a field of characteristic $\neq 2$. Then the morphism $\mathbb{W}_m \Omega_{(-)_{\text{Zar}}}^{n-1} \rightarrow \mathcal{TCH}^n(-, n; m)_{\text{Zar}}$ of §3.4 is an isomorphism of Zariski sheaves on \mathbf{Reg}_k .*

If k has characteristic $p > 2$, then the morphism $W \Omega_{(-)_{\text{Zar}}}^{n-1}/p^i \rightarrow \mathcal{TCH}_{(p)}^n(-, p^i)_{\text{Zar}}$ is an isomorphism of Zariski sheaves on \mathbf{Reg}_k for $0 \leq i \leq \infty$.

13. APPLICATIONS

In this section, we discuss some applications of Theorem 4.2.2. We let k be a field of characteristic $\neq 2$.

13.1. Gersten conjecture for additive higher Chow groups. Let X be a regular scheme over a field k . For a presheaf F of abelian groups defined on a suitable subcategory of \mathbf{Sch}_k , the Gersten conjecture for the functor F is whether the Cousin complex of F is exact. Such results were proven for the higher algebraic K -theory on \mathbf{Reg}_k by Quillen [40, Theorem 5.11] using a sort of presentation lemma, and since then it is known that various functors satisfy the Gersten conjecture. For instance, it was proven for Milnor K -theory (see [24]) and for the de Rham-Witt complex (see [13]). We have the following answer for additive higher Chow groups:

Theorem 13.1.1. *When X is a regular scheme over a field k of characteristic $\neq 2$, the Gersten conjecture holds for additive higher Chow presheaves on X in the Milnor range. More precisely, the following Cousin complex on X_{Zar} is exact after Zariski sheafifications:*

$$0 \rightarrow \mathcal{TCH}^n(-, n; m)|_X \rightarrow \mathcal{TCH}^n(K, n; m) \rightarrow \coprod_{x \in X^{(1)}} (i_x)_* H_x^1(\mathcal{TCH}^n(-, n; m)) \rightarrow \coprod_{x \in X^{(2)}} (i_x)_* H_x^2(\mathcal{TCH}^n(-, n; m)) \rightarrow \cdots$$

where $K = k(X)$, and $X^{(i)}$ is the set of codimension i points. In particular, for any point $\mathfrak{p} \in X$, the natural map $\mathcal{TCH}^n(\mathcal{O}_{X, \mathfrak{p}}, n; m) \rightarrow \mathcal{TCH}^n(K, n; m)$ is injective.

Proof. This follows from Theorem 4.2.2, because Corollary 12.2.3 implies that we have an isomorphism of Zariski sheaves $\mathcal{TCH}^n(-, n; m)_{\text{Zar}} \simeq \mathbb{W}_m \Omega_{(-)_{\text{Zar}}}^{n-1}$ on \mathbf{Reg}_k , and the Cousin complex of the big de Rham-Witt complex on a regular scheme is a flasque resolution. In characteristic 0, this is just a direct product of Cousin flasque resolutions of the absolute Kähler differential forms $\Omega_{X/\mathbb{Z}}^{n-1}$. In characteristic $p > 2$, when k is perfect, it is just a direct product of Cousin flasque resolutions of p -typical de Rham-Witt forms $W_i \Omega_X^{n-1}$ ([13, Proposition 5.1.2]). When k is not perfect and R is the local ring of any point on X , we can repeat the proof of Proposition 4.2.1 to find a sequence of flat maps of regular local rings $\{R_i\}$, which are all essentially of finite type over \mathbb{F}_p such that $\varinjlim_i R_i = R$. Since the maps $R_i \rightarrow R$ are all flat, any ideal of R of a given height is an extension of ideals of the same height from R_i for all large $i \gg 0$. Since the homology commutes with the direct limit, it follows that the Cousin resolution of $\mathcal{TCH}^n(R, n; m)$ is the direct limit of the Cousin resolutions of $\{\mathcal{TCH}^n(R_i, n; m)\}$. We conclude from the case of perfect fields that the Cousin resolution of $\mathcal{TCH}^n(R, n; m)$ is exact. \square

Remark 13.1.2. Using a Gabber-style presentation lemma, in [7, Corollary 6.2.4] it was proven that any reasonable cohomology functors satisfying the Nisnevich descent property (COH1) and the projective bundle formula (COH5) must satisfy the Gersten conjecture. By [27, Theorem 3.2] (see also [26, Theorem 5.6]), we have the latter for additive higher Chow groups, but the authors yet do not know if the Nisnevich descent property holds.

13.2. Milnor K -theory and de Rham-Witt forms. As a consequence of Theorem 4.2.2, we can connect the big de Rham-Witt complex of regular semi-local k -algebras with of their Milnor K -theory in the following way. Recall from [10, §2] and [44] that there is a *Milnor-Chow homomorphism* $\phi_n^R : K_n^M(R) \rightarrow \mathrm{CH}^n(R, n)$ given by $\phi_n^R(\{b_1, \dots, b_n\}) = W_{\underline{b}}$, where $W_{\underline{b}} = \mathrm{Spec} \left(\frac{R[y_1, \dots, y_n]}{(y_1 - b_1, \dots, y_n - b_n)} \right)$. Set $\psi_{n,m}^R = \tau_{1,m}^R \otimes \phi_{n-1}^R : \mathbb{W}_m(R) \otimes_{\mathbb{Z}} K_{n-1}^M(R) \rightarrow \mathrm{TCH}^1(R, 1; m) \otimes_{\mathbb{Z}} \mathrm{CH}^{n-1}(R, n-1)$. Let $\xi_{n,m}^R : \mathrm{TCH}^1(R, 1; m) \otimes_{\mathbb{Z}} \mathrm{CH}^{n-1}(R, n-1) \rightarrow \mathrm{TCH}^n(R, n; m)$ be the cap product of Proposition 2.4.7.

Theorem 13.2.1. *Let k be a field of characteristic $\neq 2$ and let R be a regular semi-local k -algebra essentially of finite type. Then the assignment $(a, b_1, \dots, b_{n-1}) \mapsto [a]d\log[b_1] \wedge \dots \wedge d\log[b_{n-1}]$ defines a map $\theta_{n,m}^R : \mathbb{W}_m(R) \otimes_{\mathbb{Z}} K_{n-1}^M(R) \rightarrow \mathbb{W}_m\Omega_R^{n-1}$ such that the diagram*

$$(13.1) \quad \begin{array}{ccc} \mathbb{W}_m(R) \otimes_{\mathbb{Z}} K_{n-1}^M(R) & \xrightarrow{\psi_{n,m}^R} & \mathrm{TCH}^1(R, 1; m) \otimes_{\mathbb{Z}} \mathrm{CH}^{n-1}(R, n-1) \\ \theta_{n,m}^R \downarrow & & \downarrow \xi_{n,m}^R \\ \mathbb{W}_m\Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m) \end{array}$$

commutes.

Proof. Recall that $K_*^M(R)$ is the quotient of the tensor algebra $T_*(R^\times)$ over \mathbb{Z} by the two-sided ideal generated by $\{a \otimes b \mid a, b \in R^\times, a + b = 0, 1\}$. In any case, we have a diagram

$$(13.2) \quad \begin{array}{ccc} \mathbb{W}_m(R) \otimes_{\mathbb{Z}} T_{n-1}(R^\times) & \xrightarrow{\tilde{\psi}_{n,m}^R} & \mathrm{TCH}^1(R, 1; m) \otimes_{\mathbb{Z}} \mathrm{CH}^{n-1}(R, n-1) \\ \tilde{\theta}_{n,m}^R \downarrow & & \downarrow \xi_{n,m}^R \\ \mathbb{W}_m\Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m), \end{array}$$

where $\tilde{\theta}_{n,m}^R(x \otimes (b_1 \otimes \dots \otimes b_{n-1})) = x d\log[b_1] \wedge \dots \wedge d\log[b_{n-1}]$. To check the diagram (13.2) commutes, we can use (3.2) and the identity $V_i([a]_{[m/i]}) = (1 - aT^i)$ to reduce to the case when $x = [a]$ for some $a \in R$. But in this case, the commutativity of (13.2) is obvious from the definitions of various arrows.

To deduce the commutativity of (13.1), we need to show that $\tilde{\theta}_{n,m}^R$ factors through the quotient $\mathbb{W}_m(R) \otimes_{\mathbb{Z}} K_{n-1}^M(R)$. Let $I_{n-1} = \ker(T_{n-1}(R^\times) \twoheadrightarrow K_{n-1}^M(R))$. Then we know from [10, Lemma 2.1] that $\tilde{\psi}_{n,m}^R$ kills $\mathbb{W}_m(R) \otimes_{\mathbb{Z}} I_{n-1}$. In particular, we get $\xi_{n,m}^R \circ \tilde{\psi}_{n,m}^R(\mathbb{W}_m(R) \otimes_{\mathbb{Z}} I_{n-1}) = 0$. Hence, $\tau_{n,m}^R \circ \tilde{\theta}_{n,m}^R(\mathbb{W}_m(R) \otimes_{\mathbb{Z}} I_{n-1}) = 0$. But, by Theorem 4.2.2, $\tau_{n,m}^R$ is injective, so that $\tilde{\theta}_{n,m}^R(\mathbb{W}_m(R) \otimes_{\mathbb{Z}} I_{n-1}) = 0$. This implies that the induced diagram (13.1) from (13.2) commutes. \square

Remark 13.2.2. We guess that the map $\theta_{n,m}^R$ is surjective, but do not yet know how to prove it.

13.3. Trace maps for big de Rham-Witt forms. In Proposition 12.1.5, we saw that for a simple extension $R \subset S$ of regular semi-local algebras essentially of finite type over a perfect field k of characteristic $\neq 2$, all forms in $\mathbb{W}_m \Omega_S^{n-1}$ are traceable to R for $m, n \geq 1$. We used it to prove Theorem 4.2.2, but this in turn allows us to construct the trace map on the big de Rham-Witt forms for all finite extensions of regular k -algebras essentially of finite type. This answers a question of L. Hesselholt:

Theorem 13.3.1. *Let k be a field of characteristic $\neq 2$. Let $R \subset R'$ be a finite extension of regular k -algebras essentially of finite type. Then there exists a trace map $\mathrm{Tr}_{R'/R} : \mathbb{W}_m \Omega_{R'}^n \rightarrow \mathbb{W}_m \Omega_R^n$. It is transitive; if $R \subset R' \subset R''$ are finite extensions of regular k -algebras essentially of finite type, then we have $\mathrm{Tr}_{R''/R} = \mathrm{Tr}_{R''/R'} \circ \mathrm{Tr}_{R'/R}$.*

Moreover, there is a commutative diagram

$$(13.3) \quad \begin{array}{ccc} \mathbb{W}_m \Omega_{R'}^{n-1} & \xrightarrow{\tau_{n,m}^{R'}} & \mathrm{TCH}^n(R', n; m) \\ \mathrm{Tr}_{R'/R} \downarrow & & \downarrow f_* \\ \mathbb{W}_m \Omega_R^{n-1} & \xrightarrow{\tau_{n,m}^R} & \mathrm{TCH}^n(R, n; m). \end{array}$$

Proof. Let $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(R')$, and $f : Y \rightarrow X$ be the associated finite map. We have maps $\mathbb{W}_m \Omega_R^{n-1} \xrightarrow{\tau_{n,m}^R} \mathrm{TCH}^n(R, n; m) \rightarrow H_{\mathrm{Zar}}^0(X, \mathcal{TCH}^n(-, n; m)_{\mathrm{Zar}})$, and similarly for R' . The push-forward on additive higher Chow groups yields the corresponding map of presheaves $f_*(\mathcal{TCH}^n(-, n; m)|_Y) \rightarrow \mathcal{TCH}^n(-, n; m)|_X$. After sheafifying and taking global sections, we get $f_* : H_{\mathrm{Zar}}^0(Y, \mathcal{TCH}^n(-, n; m)_{\mathrm{Zar}}) \rightarrow H_{\mathrm{Zar}}^0(X, \mathcal{TCH}^n(-, n; m)_{\mathrm{Zar}})$. On the other hand, we have a morphism of Zariski sheaves $\mathbb{W}_m \Omega_{(-)\mathrm{Zar}}^{n-1} \rightarrow \mathcal{TCH}^n(-, n; m)_{\mathrm{Zar}}$ on \mathbf{Sch}_k (see §3.4), which is an isomorphism on \mathbf{Reg}_k by Corollary 12.2.3.

Hence, if we let $\mathbb{W}_m \Omega_X^{n-1}$ denote the Zariski sheaf $\mathbb{W}_m \Omega_{(-)\mathrm{Zar}}^{n-1}|_X$ (similarly, for Y), then we have a push-forward $f_* : H_{\mathrm{Zar}}^0(Y, \mathbb{W}_m \Omega_Y^{n-1}) \rightarrow H_{\mathrm{Zar}}^0(X, \mathbb{W}_m \Omega_X^{n-1})$. But, the correspondence $X \mapsto \mathbb{W}_m \Omega_X^{n-1}$ is a quasi-coherent sheaf of $\mathbb{W}_m \mathcal{O}_X$ -modules (see [18, §5], for instance), so that the map $\mathbb{W}_m \Omega_R^{n-1} \rightarrow H_{\mathrm{Zar}}^0(X, \mathbb{W}_m \Omega_X^{n-1})$ is an isomorphism. Similarly, $\mathbb{W}_m \Omega_{R'}^{n-1} \rightarrow H_{\mathrm{Zar}}^0(Y, \mathbb{W}_m \Omega_Y^{n-1})$ is an isomorphism. Hence, f_* uniquely defines a map, denoted $\mathrm{Tr}_{R'/R} : \mathbb{W}_m \Omega_{R'}^{n-1} \rightarrow \mathbb{W}_m \Omega_R^{n-1}$. By construction, commutativity of the diagram (13.3) holds.

For a tower of finite extensions $R \subset R' \subset R''$, we repeat the above procedure to $g : \mathrm{Spec}(R'') \rightarrow \mathrm{Spec}(R')$ and $f \circ g : \mathrm{Spec}(R'') \rightarrow \mathrm{Spec}(R)$, and combining with the transitivity of proper push-forwards of additive higher Chow cycles, we immediately deduce that $(f \circ g)_* = f_* \circ g_*$. \square

We can generalize Theorem 13.3.1 a bit. We use the following, based on a suggestion of M. Spivakovsky:

Lemma 13.3.2. *Let $f : A \hookrightarrow B$ be a finite extension of noetherian regular semi-local rings containing a field. Then there exists a perfect field k and a direct system*

of finite extensions of smooth semi-local k -algebras $\{f_i : A_i \rightarrow B_i\}_{i \geq 0}$ essentially of finite type such that

$$\varinjlim_i A_i = A, \varinjlim_i B_i = B \text{ and } \varinjlim_i f_i = f.$$

Proof. Assume $d = \dim(A) = \dim(B)$. We can write $B = \frac{A[x_1, \dots, x_n]}{(g_1, \dots, g_m)}$. Let $A_\Sigma^{[n]}$ denote the semi-local ring which is the localization of $A[x_1, \dots, x_n]$ at the finite set of maximal ideals $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ such that their images in B are all the maximal ideals of B . Let $\mathfrak{p} = \ker(A_\Sigma^{[n]} \twoheadrightarrow B)$. Since $A_\Sigma^{[n]}$ is regular semi-local, we can write $\mathfrak{p} = (y_1, \dots, y_n)$ and $\mathfrak{m}_j = (y_1, \dots, y_n, x_1^j, \dots, x_d^j)$ for some elements y_i , where $1 \leq i \leq n$, and x_ℓ^j , where $1 \leq \ell \leq d$ and $1 \leq j \leq r$.

Now, using the Néron-Popescu desingularization (see [42, Theorem 1.1]), we can write $A = \varinjlim_i A_i$, where $\{A_i\}_{i \geq 0}$ is a direct system of smooth semi-local k -algebras essentially of finite type, and k is the prime field of A , which is clearly perfect. After leaving out some finitely many elements in the direct system from the bottom, we can assume that $\mathfrak{p}_0 := (y_1, \dots, y_n) \subset A_0^{[n]}$ so that $\mathfrak{p}_0 A^{[n]} = \mathfrak{p}$. We can also assume that $g_1, \dots, g_m \in A_0^{[n]}$. Set $\mathfrak{m}_{i,j} = \mathfrak{m}_j \cap A_i^{[n]}$ for $i \geq 1$, $1 \leq j \leq r$, $\Sigma_i = \{\mathfrak{m}_{i,1}, \dots, \mathfrak{m}_{i,r}\}$ and $B_i = (A_i^{[n]})_{\Sigma_i} / (y_1, \dots, y_n)$.

Since $A \rightarrow B$ is finite and $\mathfrak{p}_0 \subset A_0^{[n]}$, it follows that each $A_i \hookrightarrow B_i$ is a finite extension. It is left to show that each B_i is regular to prove the lemma. To show this, we only need to check that y_1, \dots, y_n form a part of a regular system of parameters in $A_{\mathfrak{m}_{i,j}}^{[n]}$ for each $1 \leq j \leq r$. Indeed, if they do not form a part of a regular system of parameters, then their images in $\mathfrak{m}_{i,j} / \mathfrak{m}_{i,j}^2$ become linearly dependent over $A_i^{[n]} / \mathfrak{m}_{i,j}$. But this implies that the images of y_1, \dots, y_n in $\mathfrak{m}_j / \mathfrak{m}_j^2$ become linearly dependent over $A^{[n]} / \mathfrak{m}_j$, which contradicts our choice of generators of \mathfrak{m}_j . This finishes the proof. \square

Theorem 13.3.3. *Let $f : R \hookrightarrow R'$ be a finite extension of noetherian regular semi-local rings containing a field of characteristic $\neq 2$. Then there exists a trace map $\text{Tr}_{R'/R} : \mathbb{W}_m \Omega_{R'}^n \rightarrow \mathbb{W}_m \Omega_R^n$. It is transitive; if $R \subset R' \subset R''$ are finite extensions of regular semi-local rings containing a field of characteristic $\neq 2$, then we have $\text{Tr}_{R''/R} = \text{Tr}_{R'/R} \circ \text{Tr}_{R''/R'}$.*

Proof. By Lemma 13.3.2, we can find a perfect field k of characteristic $\neq 2$ and a direct system of finite extensions of smooth semi-local k -algebras $\{f_i : R_i \rightarrow R'_i\}_{i \geq 0}$ essentially of finite type such that $\varinjlim_i R_i = R$, $\varinjlim_i R'_i = R'$ and $\varinjlim_i f_i = f$. This yields a diagram

$$(13.4) \quad \begin{array}{ccc} \varinjlim_i \mathbb{W}_m \Omega_{R'_i}^n & \xrightarrow{\varinjlim_i \text{Tr}_{R'_i/R_i}} & \varinjlim_i \mathbb{W}_m \Omega_{R_i}^n \\ \downarrow & & \downarrow \\ \mathbb{W}_m \Omega_{R'}^n & \dashrightarrow & \mathbb{W}_m \Omega_R^n \end{array}$$

where the top horizontal arrow is the limit of the trace maps of Theorem 13.3.1. Since the two vertical arrows are isomorphisms by [41, Proposition 1.16], we get a unique trace map $\text{Tr}_{R'/R} : \mathbb{W}_m \Omega_{R'}^n \rightarrow \mathbb{W}_m \Omega_R^n$ such that the above diagram

commutes. The transitivity of this trace map is easily proved using the transitivity of the top horizontal arrow and the transitive nature of the construction of the Néron-Popescu desingularization in Lemma 13.3.2. \square

Remark 13.3.4. In 2013, JP was told by K. Rülling that, using [8], [9] and the duality machine in [15], one may prove the existence of the trace map $\mathrm{Tr}_{R'/R}$ of Theorem 13.3.1 abstractly. For the p -typical case, the essential idea can be found in [21, Appendix B]. On the other hand, S. Kelly contacted JP that “using the ideas in [23, §3.6,3.7], one may promote a suitable structure of traces on regular dimension ≤ 1 essentially smooth k -algebras to a structure of transfers on smooth k -schemes in the sense of Voevodsky. cf. [19, Lemma 4.15].” We believe these should coincide with the one in Theorem 13.3.1, e.g. by modifying the argument of Proposition 12.1.5, one may show the uniqueness of the trace subject to (13.3).

13.4. Comparison with crystalline cohomology. Let k be a perfect field of characteristic $p > 2$. Recall that crystalline cohomology $H^*(X/W)$ of a smooth scheme X of finite type over k was defined by Berthelot [3] as the cohomology of the sheaf $\mathcal{O}_{X,\mathrm{crys}}$ on the crystalline site, as outlined by Grothendieck [14]. Bloch showed that for a smooth proper scheme of dimension $< p$, this cohomology could be described in terms of the Zariski hypercohomology of a complex of relative K -theory sheaves. With an idea of Deligne, Illusie constructed the p -typical de Rham-Witt complex whose Zariski hypercohomology is the crystalline cohomology. The following is a new description of crystalline cohomology in terms of algebraic cycles, which is a cycle-theoretic avatar of the K -theoretic description of [4]:

Theorem 13.4.1. *Let X be a smooth scheme of finite type over a perfect field k of characteristic $p > 2$. Then for $n \geq 0$, there is a canonical isomorphism*

$$(13.5) \quad H_{\mathrm{crys}}^n(X/W) \simeq \varprojlim_i \mathbb{H}_{\mathrm{Zar}}^{n+1}(X, \mathcal{TCH}_{(p)}^M(-; p^i)_{\mathrm{Zar}}).$$

Proof. It follows by Corollary 12.2.3 that the map of the complexes of Zariski sheaves $\psi_X : W\Omega_X^\bullet/p^i \rightarrow \mathcal{TCH}_{(p)}^M(X; p^i)_{\mathrm{Zar}}[1]$ on X is an isomorphism.

For $i \geq 0$, let $W_i\Omega_{X,\mathrm{DI}}^\bullet$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto W_i\Omega_{\mathcal{O}(U),\mathrm{DI}}^\bullet$, where $W_i\Omega_{R,\mathrm{DI}}^\bullet$ is the Deligne-Illusie de Rham-Witt complex of the ring R . We define the sheaf $W\Omega_{X,\mathrm{DI}}^\bullet$ similarly. It follows immediately from the definition of $W\Omega_X^\bullet$ (see § 3.4) that there is a natural map $\phi_X : W\Omega_X^\bullet \rightarrow W\Omega_{X,\mathrm{DI}}^\bullet$ of the complexes of Zariski sheaves. Moreover, it follows from [20, Proposition I.1.13.1] that ϕ_X is an isomorphism. We thus get a set of morphisms of chain complexes of Zariski sheaves:

$$(13.6) \quad \begin{array}{ccc} W\Omega_X^\bullet/p^i & \xrightarrow{\phi_X} & W\Omega_{X,\mathrm{DI}}^\bullet/p^i \\ \psi_X \downarrow & & \downarrow \tau_X \\ \mathcal{TCH}_{(p)}^M(X; p^i)_{\mathrm{Zar}}[1] & \leftarrow \cdots \leftarrow & W_i\Omega_{X,\mathrm{DI}}^\bullet \end{array}$$

Here, τ_X is a natural map of chain complexes which is a quasi-isomorphism by [20, Corollaire I.3.17]. We conclude that $\psi_X \circ \phi_X^{-1} \circ \tau_X^{-1} : W_i\Omega_{X,\mathrm{DI}}^\bullet \rightarrow \mathcal{TCH}_{(p)}^M(X; p^i)_{\mathrm{Zar}}[1]$ is an isomorphism in $\mathcal{D}^+(X_{\mathrm{Zar}})$. In particular, the induced map on the hypercohomology is an isomorphism. Taking the limit, we get (13.5). \square

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